

PRELIMINARY ANALYSIS OF VARIATION OF PITCH MOTION OF A  
VEHICLE IN A SPACE ENVIRONMENT DUE TO FUEL SLOSHING  
IN A RECTANGULAR TANK

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SUMMARY

The equation of motion is derived for the angular rotation about the pitch axis of a space vehicle that is thrusting while partly filled with a liquid fuel that constitutes a large percentage of its mass. The forces due to thrust and rotation are of a magnitude equal to the capillary forces of the fluid and much greater than the Coriolis force. These forces are idealized to lie parallel to a longitudinal axis of the fluid tank. Changes in the angular rotation are generated by an impulsive torque applied to the vehicle. The fluid is considered to be irrotational, incompressible, inviscid, and contained in a rectangular tank.

The resulting angular rotation of the vehicle and fluid system consists of the sum of a constant or average rotation and a cosine variation with time due to the motion of the fluid within the tank. The frequency of the variation is the slosh frequency of the free surface of the fluid. The slosh frequency is presented as a function of the inertial loading due to thrust, rotation, the amount of fluid in the tank, and the fluid-surface tension and density. The limit of thrust reversal for stability of the free-surface motion is established from the slosh frequency. The ability to minimize the sloshing effect by proper adjustment of the impulse magnitude and time of application is discussed and an example is given. The response per unit impulse of a vehicle is presented for different loading conditions and fuel available for the change in angular velocity at the time of impulse, the average angular rate, the amplitude of the slosh variation, and the ratio of slosh variation to average angular rate.

INTRODUCTION

Vehicles operating in space can require maneuvers such as maintaining a fixed attitude or rendezvous, where associated inertial forces on the fluid are equal in magnitude to the capillary forces. As these space vehicles become larger and more complex, it is expected that they will be carrying a large percentage of their total mass in liquid fuel. Thus, as the liquid fuel is used, a free surface is created that is put in motion by the changes in the

gravitational field due to changes in thrust level or angular rotation in an orientation maneuver. The motion of the fluid would appear to cause a variation in the motion of the vehicle at a time when it is desired that the motion be as constant as possible. Some work has been done on the problem of fuel sloshing in an environment where the inertial forces are much greater than the capillary forces of the fluid. (For example, see ref. 1.) In addition, in the area where the inertial forces are equal to the capillary forces, the frequency of surface oscillation or slosh frequency has been determined. (See refs. 2 to 5.) However, none of the aforementioned work has included the velocity of the fluid and the resulting effect on the motion of the vehicle in a space environment.

This report is a preliminary analytical investigation of the problem of pitch-motion variation of a vehicle due to the motion of the liquid fuel relative to the tank, under low inertial forces created by thrust and angular rotation. The fuel is considered to be irrotational, incompressible, inviscid, and contained in a rectangular tank. As a first step in the analysis, a linearized expression for the sloshing frequency of a fluid free surface is derived. The frequency is obtained by assuming small amplitude displacements about a flat equilibrium free surface, an average angular rotation, and no variation in the amount of the fluid during thrust. The frequency is shown to be a function of the acceleration loading, surface tension, and centrifugal force due to angular rotation about the pitch axis of a tank containing a specified amount of a given fluid. The equation of motion for the angular velocity of the vehicle and fluid is then derived for motion started by an impulsive torque. In general, the angular motion consists of a constant angular rotation with a superimposed cosine variation with time. The possibility of minimizing the disturbance due to sloshing by selecting the proper time of impulse application as a function of the slosh frequency is investigated. Inasmuch as the time required for such a maneuver can be excessive for large tanks and low frequencies, the use of this maneuver is limited.

In this report, the margin for stable motion of the liquid free surface is shown. The variation of the slosh frequency is presented as a function of the ratios of fluid mass to tank mass for various initial tank capacities and different loading conditions. The response from rest to a unit impulsive torque is shown for a particular vehicle containing various amounts of liquid oxygen, in different-size tanks, and under different loading conditions.

## SYMBOLS

$A_N, B_N, C_N, D_N$	constants in general solution for $\nabla^2 \phi$
$a$	tank-carried acceleration field due to thrust, $\text{ft}/\text{sec}^2$
$d_N, e_N$	coefficients of $E_N$ in equation (10), $\text{ft}^2/\text{sec}$
$E_N$	time variation of $\phi$ for odd integer in equation (8)

$F_N$	time variation of $\phi$ for even integer in equation (8)
$F_b$	body force on fluid particle
$f$	constant defined by equation (21), slugs
$f_r$	tank fineness ratio, $L/R$
$G$	constant defined by equation after equation (15), $ft^2$
$H$	total angular momentum, $H_t + H_f$ , slug- $ft^2/sec$
$H_t$	tank angular momentum, slug- $ft^2/sec$
$H_f$	fluid angular momentum, slug- $ft^2/sec$
$h$	height of equilibrium fluid free surface above tank base, ft
$\underline{h} = h/R$	
$I$	total system inertia about y-axis, slug- $ft^2$
$I_f$	moment of inertia of fluid about y-axis, slug- $ft^2$
$I_t$	moment of inertia of tank about y-axis, slug- $ft^2$
$i, j, k$	unit vectors directed along x-, y-, and z-axes, respectively
$K$	mean curvature of free surface, $\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$ , $ft^{-1}$
$L$	tank length, ft
$m_f$	fluid mass, slugs
$m_t$	inertial tank mass, $I_t/R^2$ , slugs
$N$	an integer in infinite summation for $\phi$
$N_i$	impulsive torque, ft-lb
$N_{Bo}$	Bond number, $\rho R^2 a / \tau$
$N_{ce}$	centrifuge number, $\rho R^3 \underline{h} \omega^2 / \tau$
$n$	unit normal to rigid boundary

P	fluid pressure, lb/sq ft
P <sub>i</sub>	impulsive pressure, lb/sq ft
P <sub>v</sub>	vapor pressure, lb/sq ft
Q	constant defined by equation after equation (22)
q	velocity relative to inertial frame, ft/sec
q <sub>t</sub>	fluid velocity relative to tank, ft/sec
R	tank half-width, ft
R <sub>1</sub> , R <sub>2</sub>	principal radii of curvature, ft
r	distance from origin to point in fluid $(x^2 + y^2 + z^2)^{1/2}$ , ft
S	fluid free surface
s	wetted surface of rigid tank wall
t	time
t <sub>1</sub>	time of removal of impulse
V	tank velocity relative to inertial-axis system $\vec{\omega} \times \vec{r}$ , ft/sec
W	thickness of tank, ft
X, Y, Z	inertial-axis system
x, y, z	tank-fixed axis system
$\alpha$	ratio of residual fluid mass to inertial tank mass
$\alpha_{ft}$	ratio of initial full-tank fluid mass to tank mass
$\beta$	ratio of fluid to tank inertia, $\frac{\alpha}{3}(1 + f_r^2 + f_r \underline{h} + \underline{h}^2)$
$\gamma$	impulse, slug-ft <sup>2</sup> /sec
$\Delta$	incremental change
$\eta = \eta(x, t)$	small displacement from equilibrium surface
$\mu = \frac{\alpha}{3}(f_r^2 + f_r \underline{h} + \underline{h}^2)$	

$\nu$	coefficient defined by equation (27b)
$\xi$	coefficient defined by equation (27a)
$\rho$	fluid density, slugs/cu ft
$\sigma_N$	Nth mode fluid slosh frequency, radians/sec
$\tau$	fluid surface tension, lb/ft
$\phi$	velocity potential function, ft <sup>2</sup> /sec
$\psi$	phase angle in system motion, arc tan $\frac{\nu}{\xi}$
$\Omega$	body force potential
$\omega$	tank angular motion, radians/sec
$\bar{\omega}$	average tank angular motion defined by equation (26), radians/sec
$\omega_s$	slosh coefficient defined by equation (27), radians/sec
$\nabla$	vector operator, $\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}$

$(\vec{r}) \times (\vec{r})$  vector cross product

$(\vec{r}) \cdot (\vec{r})$  vector dot product

Subscripts:

$\left| \begin{array}{l} \\ F \end{array} \right.$  relative to fixed coordinate system

$\left| \begin{array}{l} \\ M \end{array} \right.$  relative to moving coordinate system

1 relative to first mode

o relative to conditions at time of last impulse

W at wall  $x = -r$

Dots over symbols indicate derivatives with respect to time.

Arrows over symbols indicate vectors.

A prime denotes conditions prior to last impulse.

## THEORY

### Derivation of General Bernoulli Equation

As seen in figure 1, the cross section of a rectangular tank with length  $L$ , half-width  $R$ , thickness  $W$ , and containing a fixed amount of an inviscid, incompressible fluid rotates slowly about an inertial axis fixed in the base of the tank. The origin of this axis coincides with the center of gravity of the vehicle so that the inertial-axis system  $(X,Y,Z)$  and a tank-fixed coordinate system  $(x,y,z)$  is formed, with the  $Y,y$ -axis being the axis of rotation. While the tank rotates about this pitch axis, the tank and fluid are subjected to an acceleration field directed along the positive  $z$ -axis. This situation corresponds to a spacecraft thrusting while changing its spatial orientation. Therefore, because of this longitudinal and centrifugal loading, the liquid is positioned in the upper portion of the tank.

The equation of motion of an inviscid, incompressible fluid particle relative to the inertial system is

$$\left. \frac{d\vec{q}}{dt} \right|_F = \left. \frac{d\vec{q}}{dt} \right|_M + \vec{\omega} \times \vec{q} = \vec{F}_b - \frac{\nabla p}{\rho} \quad (1)$$

where the velocity of the particle is

$$\vec{q} = \vec{q}_t + \vec{V}$$

and where

$\vec{q}_t$	velocity vector of the fluid particle relative to tank
$\vec{V}$	velocity vector of tank
$p$	fluid pressure
$\rho$	fluid density
$\vec{F}_b$	body force
$\left _F\right.$	relative to the fixed coordinates
$\left _M\right.$	relative to the moving coordinates

The body forces are assumed to be derivable from a scalar potential as follows:

$$\vec{F}_b = -\nabla\Omega$$

Substituting for  $\vec{q}$  from equation (1) and taking the time derivative gives

$$\begin{aligned}\left.\frac{d\vec{q}}{dt}\right|_F &= \left.\frac{d}{dt}(\vec{q}_t + \vec{\omega} \times \vec{r})\right|_M + \vec{\omega} \times (\vec{q}_t + \vec{\omega} \times \vec{r}) \\ &= \frac{\partial \vec{q}_t}{\partial t} + \frac{\partial}{\partial t}(\vec{\omega} \times \vec{r}) + (\vec{q}_t \cdot \nabla)(\vec{q}_t + \vec{\omega} \times \vec{r}) + \vec{\omega} \times \vec{q}_t + \vec{\omega} \times (\vec{\omega} \times \vec{r})\end{aligned}$$

Since the fluid motion relative to the tank is considered irrotational,

$$\vec{q}_t|_M = -\nabla\phi$$

where the velocity potential  $\phi$  is a function of the spatial coordinates and time. Also,

$$\nabla \times \vec{q}_t = 0$$

and

$$\nabla^2\phi = 0$$

Therefore, by substitution,

$$\left.\frac{d\vec{q}}{dt}\right|_F = -\frac{\partial(\nabla\phi)}{\partial t} + \dot{\vec{\omega}} \times \vec{r} - 2\vec{\omega} \times \nabla\phi + \nabla \frac{\nabla\phi \cdot \nabla\phi}{2} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) - (\nabla\phi \cdot \nabla)\vec{\omega} \times \vec{r}$$

By expanding the last two terms and substituting the acceleration into the general equation of motion, the following pressure equation is obtained:

$$\nabla \left( -\frac{\partial\phi}{\partial t} + \frac{\nabla\phi \cdot \nabla\phi}{2} + \Omega + \frac{p}{\rho} - \frac{r^2\omega^2}{2} \right) + \vec{\omega}(\vec{\omega} \cdot \vec{r}) = -\dot{\vec{\omega}} \times \vec{r} + 3\vec{\omega} \times \nabla\phi \quad (2)$$

The problem is now restricted to the case where the motion is considered to be in the X,Z plane and about the Y-axis. The following terms are assumed to be negligible to allow a solution of  $\phi$  by the separation of variables and to simplify the analysis:

$$(I) \quad \nabla\phi \cdot \nabla\phi$$

$$(II) \quad \vec{\omega} \times \nabla\phi$$

$$(III) \quad \dot{\vec{\omega}} \times \vec{r}$$

Assumption (I) implies that  $|\nabla\phi|$  is very small. The angular rate is also considered small, but has an order of magnitude greater than  $|\nabla\phi|$ .

Since  $|\vec{\omega} \times \nabla\phi|$  is of the order of  $|\omega\nabla\phi|$  and if the tank dimensions are large, then assumption (II) implies that

$$\left| \nabla \left( \frac{r^2 \omega^2}{2} \right) \right| \gg |\omega \nabla \phi|$$

or

$$|r\omega| \gg |\nabla\phi|$$

Physically, this relationship means that the velocity of the particle considered as fixed to the tank is much greater than the velocity of the particle relative to the tank and that the centrifugal force on the particle is much greater than the Coriolis force.

The implication from assumption (III) is that the rotational acceleration is negligible and that there are no external forces acting on the particle during the period to be analyzed. This assumption will also be used later in obtaining an expression for the free-surface motion. With these assumptions and relationships and by representing the body force in terms of the magnitude of the carried vehicle acceleration field "a" due to thrust along the longitudinal z-axis so that  $\Omega = -az$ , equation (2) is integrated to yield the general Bernoulli pressure equation:

$$\frac{\partial\phi}{\partial t} + \frac{r^2 \omega^2}{2} + az = \frac{p}{\rho}$$

The term containing  $\omega^2$  represents the velocity squared of the particle. By assuming that the fluid is at a distance from the tank base that is greater than the tank half-width, the term can be linearized as follows:

$$\frac{r^2 \omega^2}{2} \approx \frac{z^2 \omega^2}{2}$$

This approximation improves as the fineness ratio increases. Therefore, the pressure equation may be written as

$$\frac{\partial\phi}{\partial t} + \frac{z^2 \omega^2}{2} + az = \frac{p}{\rho} \quad (2a)$$

where the constant of integration is set equal to zero without loss of generality (ref. 2).

### Rigid-Wall Boundary Conditions

In order to determine the function  $\phi$  in spatial coordinates, it is only necessary to prescribe conditions along the boundaries. The function  $\phi$  is then determined completely throughout the fluid (ref. 3).

The following condition for the velocity of the particle evaluated at and normal to the rigid walls must also hold:

$$\nabla\phi \cdot \vec{n} = 0$$

which means for the tank under consideration

$$\left. \begin{aligned} \frac{\partial\phi}{\partial x} \Big|_{x=\pm R} &= 0 \\ \frac{\partial\phi}{\partial z} \Big|_{z=L} &= 0 \end{aligned} \right\} \quad (3)$$

Equations (3) indicate that the only direction the particle can move at the wall is parallel to the wall. The remaining boundary is along the free surface. The free surface consists of small displacements about an equilibrium configuration.

### Free-Surface Conditions

In references 5 and 6 the contact angle, which is the intersection between the liquid, gas, and rigid tank wall, is shown to have a large influence on the equilibrium free surface with a flat surface obtained for a lower range of loading for a contact angle of  $90^\circ$ . In addition, the contact angle does not change with the loading. Since the pressure due to the centrifugal force has been idealized to be parallel to the  $\vec{a}$  vector, the free surface is assumed to be flat throughout the range of loading. The free surface under these conditions may be simply defined as

$$S = z - [h + \eta(x,t)] = 0 \quad (4)$$

where  $h$  is the height of the equilibrium free surface and  $\eta(x,t)$  is a small perturbation about the equilibrium surface. The height of the equilibrium free surface is considered constant throughout the maneuvering.

With the fluid particles remaining at the surface, the boundary condition for the free surface is (ref. 4)

$$\frac{DS}{Dt} = 0$$

where  $DS/Dt$  is the total time derivative of  $S$ . By substituting  $\vec{q}_t = -\nabla\phi$

$$\frac{\partial S}{\partial t} - \left( \frac{\partial S}{\partial z} \frac{\partial \phi}{\partial z} \right) - \left( \frac{\partial S}{\partial y} \frac{\partial \phi}{\partial y} \right) - \left( \frac{\partial S}{\partial x} \frac{\partial \phi}{\partial x} \right) = 0$$

Substituting for  $S$  from equation (4) yields the following equation for the free-surface boundary condition with the velocity component in the x-direction considered negligible (ref. 3):

$$\frac{\partial \eta(x,t)}{\partial t} = - \frac{\partial \phi}{\partial z} \quad (5)$$

Hereinafter, the displacement from the equilibrium surface  $\eta(x,t)$  is designated  $\eta$ .

In order to obtain the effect of the free-surface boundary condition on the motion of the particle, equation (2a) is utilized. Although the equilibrium free surface is assumed to be unaffected by the surface tension, the motion of the fluid about the equilibrium position is affected. Subtracting the constant vapor pressure of the fluid from both sides of equation (2a) leads to:

$$\frac{\partial \phi}{\partial t} + \frac{z^2 \omega^2}{2} + az = \frac{\tau K}{\rho} + \frac{p_v}{\rho} \quad (6)$$

where

$$\frac{\tau K}{\rho} = \frac{p - p_v}{\rho}$$

and where

$p_v$           vapor pressure

$\tau$             surface tension

$K$             mean curvature of free surface,  $\left( \frac{1}{2} \right) \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$

$R_1, R_2$       principal radii of curvature

Equation (6) is evaluated along the free surface as defined by equation (4). The exception to this evaluation is that the function  $\phi$  is evaluated along the equilibrium surface

$$z = h$$

This exception is due to (see ref. 7)

$$\phi(\eta) \approx 0$$

By substituting the value for  $z$  from equation (4) into equation (6) and neglecting the square of the displacement from the equilibrium surface since it is small, the equation for the perturbed free surface is

$$\frac{\partial \phi}{\partial t} + \frac{(h^2 + 2h\eta)\omega^2}{2} + a(h + \eta) = \frac{\tau K}{\rho} + \frac{P_v}{\rho} \quad (6a)$$

Taking the partial derivative of equation (6a) with respect to time and substituting the relationship for  $\eta$  from equation (5) yields

$$\frac{\partial^2 \phi}{\partial t^2} - (h\omega^2 + a)\frac{\partial \phi}{\partial z} = \frac{\tau}{\rho} \frac{\partial K}{\partial t} \quad (7)$$

where the initial assumption  $\dot{\omega} \approx 0$  has been maintained.

The mean curvature for a two-dimensional surface is defined, at an instant of time, as

$$K = \frac{\partial^2 z / \partial x^2}{\left[ 1 + \left( \frac{\partial z}{\partial x} \right)^2 \right]^{3/2}}$$

By substituting the value for  $z$  from equation (4) and neglecting the square of the small displacement term, the partial derivative with respect to time of the mean curvature reduces to

$$\frac{\partial K}{\partial t} = \frac{\partial^3 \eta}{\partial x^2 \partial t} = \frac{-\partial^3 \phi}{\partial x^2 \partial z}$$

Substituting this value for  $\partial K / \partial t$  into equation (7) yields the following equation for the free-surface boundary condition:

$$\frac{\partial^2 \phi}{\partial t^2} - (h\omega^2 + a)\frac{\partial \phi}{\partial z} + \frac{\tau}{\rho} \frac{\partial^3 \phi}{\partial x^2 \partial z} = 0 \quad (7a)$$

#### Space-Coordinate Solution of Velocity Potential Function

Equation (6) presents the derived equation of motion for the free surface of the fluid. Equations (7a) and (3) give the boundary conditions along the free surface and rigid walls, respectively. The velocity potential function  $\phi$  must satisfy these equations. Since the fluid is considered to be incompressible and inviscid and the particle motion irrotational, the velocity potential function must also satisfy Laplace's equation throughout the fluid:

$$\nabla^2 \phi = 0$$

Utilizing separation of variables and applying the boundary conditions along the rigid tank walls (eqs. (3)) yields a general solution for  $\phi$ :

$$\phi = \sum_{N=1}^{\infty} \left( A_N \cos \frac{N\pi x}{2R} + B_N \sin \frac{N\pi x}{2R} \right) \left( C_N \sinh \frac{N\pi z}{2R} + D_N \cosh \frac{N\pi z}{2R} \right)$$

Thus, the solution of  $\phi$  depends upon whether  $N$  is odd or even; that is,

$$\phi = \sum_{N=\text{Odd}} E_N \sin \frac{N\pi x}{2R} \cosh \frac{N\pi(L-z)}{2R} + \sum_{N=\text{Even}} F_N \cos \frac{N\pi x}{2R} \cosh \frac{N\pi(L-z)}{2R} \quad (8)$$

The constants  $E_N$  and  $F_N$  are functions of time and are the result of a combination of the original constants in the general expression for  $\phi$ . Once  $\phi$  is solved in terms of space coordinates, the time dependency still remains to be determined. In order to determine the nature of the time dependent function in the expression for  $\phi$ , equation (8) is substituted into equation (7a). This substitution yields the following relationship for the  $N$ th mode (when  $N$  is odd):

$$\ddot{E}_N + \sigma_N^2 E_N = 0$$

An identical equation in terms of  $F_N$  can be written for  $N$  is even. Thus, the free surface has the characteristic of harmonic motion with the characteristic slosh frequency for the  $N$ th mode:

$$\sigma_N = \left\{ \left[ h\bar{\omega}^2 + a + \frac{\tau(N\pi)^2}{4\rho R^2} \right] \frac{N\pi}{2R} \tanh \frac{N\pi(L-h)}{2R} \right\}^{1/2} \quad (9)$$

The  $\bar{\omega}$  appearing in the expression for the frequency is considered as a constant or average rotational rate throughout the period of free rotation. The determination of  $\bar{\omega}$  is presented in a subsequent section of this paper.

The expression for the  $N$ th slosh frequency is generalized by two loading parameters, the centrifuge number  $N_{ce}$  and the Bond number  $N_{Bo}$ , to yield

$$\sigma_N = \left\{ \left[ N_{ce} + N_{Bo} + \frac{(N\pi)^2}{4} \right] \frac{\tau}{2\rho} \frac{N\pi}{R^3} \tanh \frac{N\pi(f_r - \underline{h})}{2} \right\}^{1/2} \quad (9a)$$

where

$f_r$  fineness ratio,  $L/R$

$$\frac{h}{R} = \frac{h}{R}$$

The centrifuge number is hereby defined as

$$N_{ce} = \frac{\rho R^3 \omega^2}{\tau}$$

and the Bond number is

$$N_{Bo} = \frac{\rho R^2 a}{\tau}$$

The expression for  $\sigma_N$  reveals that the free-surface motion becomes unstable when

$$-N_{Bo} > N_{ce} + \frac{(N\pi)^2}{4}$$

The solution for the stable harmonic motion is

$$E_N = d_N \cos \sigma_N t + e_N \sin \sigma_N t \quad (10)$$

A parallel solution can be obtained for  $F_N$ .

#### Initial Conditions

In order to define the initial conditions ( $t = 0$ ), an impulsive torque is considered to be applied to the tank and fluid system to reorient the vehicle. The magnitude of the force creating the impulsive torque is considered to be large so that the centrifugal force due to the rotation and the linear thrusting of the vehicle are neglected during the time of the impulsive torque. (See ref. 8.) As shown in reference 7 the fluid in the tank resists the impulsive torque through a system of pressures acting along the tank wall about the base point. In the present analysis, the time integral of pressure is called the impulsive pressure, and the time integral of the impulsive torque is called the impulse.<sup>1</sup>

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<sup>1</sup>For correlation of terms used herein with those in reference 7, the impulsive torque is the couple in the term "force wrench." The time integral of the impulsive torque is the impulse wrench.

It is also shown in reference 7 that when the impulse

$$\gamma = \lim_{t_1 \rightarrow t_0} \int_{t_0}^{t_1} \vec{N}_1 dt \quad (11)$$

overcomes the impulsive pressures acting on the walls, there is a net change in momentum of the tank and fluid system. The torque is idealized so that it occurs only in the x-direction and applied therefore to the tank wall  $x = -R$ , with the component over the top of the tank considered negligible. Even though the torque cannot be represented by a scalar force potential because

$$\nabla \times \vec{N}_1 \neq 0$$

it is maintained as an approximation of the physical situation. Therefore, for the net change in momentum

$$\gamma - \int_h^L \vec{r} \times \vec{n}(p_1) dz = \Delta H \quad (12)$$

where

$\vec{n}$  unit vector normal to and outward from fluid

$p_1$  impulsive pressure along wall  $x = -R$

$\Delta H$  change in angular momentum of system

The sudden motion of the walls due to the impulsive torque is considered as a series of pistons applied to the fluid surface  $x = -R$  with a force proportional to  $\dot{\omega}$  and  $z$  in a manner analogous to the procedure in reference 3. The total integral of these pistons over the wall is then equivalent to the moment acting on the fluid.

The total rate of change across the boundary  $x = -R$  may be approximated by an average or constant force on the boundary. However, since the magnitude of the impulse distribution cannot really be determined, the total effect on the fluid motion over the entire wall must be used. Thus, a correlation is obtained between the total impulsive torque and the resisting moment due to the system of impulsive pressures. When dealing in terms of the total flux across the boundary, there is no difference in the effect regardless of how the pressure is assumed to be distributed. The pressure at a point along the wall due to the impulsive torque is

$$p_w = 2\rho R z \dot{\omega}$$

For the time of the impulsive-torque application, the pressure equation can be written as

$$\left. \frac{\partial \phi}{\partial t} \right|_{x=-R} + \frac{z^2 \omega^2}{2} + az = \frac{p + p_W}{\rho}$$

Integrating this equation over the small interval of time of the impulsive-torque application, with the value of  $p_W$  much greater than the static pressure existing prior to impulse, gives

$$\phi - \phi_0 = 2Rz \Delta\omega = p_i \quad (13)$$

where  $\phi_0$  is the velocity potential function just prior to the impulsive torque. In order to find the cumulative effect of the impulse, the moments due to the impulsive pressures are integrated over the wetted tank surface  $x = -R$  with the lower limit of integration taken at the undisturbed surface:

$$\int_s \vec{r} \times \vec{n} (\phi - \phi_0) \Big|_{\substack{x=-R \\ t=0}} ds = \int_s \vec{r} \times \vec{n} (2Rz \Delta\omega) ds \quad (14)$$

where

$$ds = dy dz$$

By substituting into equation (14) the value for  $\phi$  from equation (8) and integrating, the following expression for the amplitude  $d_1$  is obtained at  $t = 0$  when only the first mode ( $N = 1$ ) is considered:

$$d_1 = -G \Delta\omega + E_1' \quad (15)$$

where

$$G = \frac{\pi^2 (L^3 - h^3)}{6R \left[ \cosh \frac{\pi(L-h)}{2R} + \frac{\pi h}{2R} \sinh \frac{\pi(L-h)}{2R} - 1 \right]}$$

The value of  $E_1'$  can be determined by the fluid conditions of velocity and surface displacement in the cycle prior to the impulse. Now, there are two unknowns,  $\Delta\omega$  in equation (15) and  $e_N$  for  $N = 1$  in equation (10). However,  $e_1$  is readily determined from the displacement of the free surface at  $t = 0$  by assuming that there is no change in spatial coordinates during the impulsive torque. The free-surface displacement is

$$\eta \Big|_{t=0} = \eta' \Big|_{t=t_0}$$

where  $\eta \Big|_{t=t_0}$  is the displacement existing at the initial time of impulsive-torque application. The relationship in equation (5) is utilized to obtain the value of  $\eta$  at  $t = 0$  from the previous potential function, which is assumed to have the same spatial boundary and loading conditions:

$$\eta \Big|_{t=0} = - \int_{z=h}^t \frac{\partial \phi_0}{\partial z} dt \Big|_{t=t_0} = \frac{\pi (d_1' \sin \sigma_1' t_0 - e_1' \cos \sigma_1' t_0)}{2R\sigma_1'} \left[ \sin \frac{\pi x}{2R} \sinh \frac{\pi(L-h)}{2R} \right]$$

Solving for  $e_1$  gives

$$e_1 = - \frac{\sigma_1}{\sigma_1'} (d_1' \sin \sigma_1' t_0 - e_1' \cos \sigma_1' t_0) \quad (16)$$

The velocity potential function may therefore be written in terms of  $\Delta\omega$  and the conditions at the time previous to the last impulse as follows:

$$\phi = \left[ (-G \Delta\omega + E_1') \cos \sigma_1 t - \frac{\sigma_1}{\sigma_1'} (d_1' \sin \sigma_1' t_0 - e_1' \cos \sigma_1' t_0) \sin \sigma_1 t \right] \sin \frac{\pi x}{2R} \cosh \frac{\pi(L-z)}{2R} \quad (17)$$

In order to obtain  $\Delta\omega$  in terms of the known impulse  $\gamma$ , equation (12) is utilized. The change in angular momentum of the system is

$$\Delta \vec{H} = I_t \Delta \vec{\omega} + \Delta \vec{H}_f$$

where  $I_t$  is the vehicle yaw moment of inertia considered as being that of the tank and  $\Delta \vec{H}_f$  is the change in angular momentum of the fluid, that is,

$$\Delta \vec{H}_f = \rho \int_{-W/2}^{W/2} \int_{-R}^R \int_h^L \vec{r} \times \left[ \Delta \vec{\omega} \times \vec{r} - \nabla (\phi - \phi_0) \right] dz dx dy \quad (18)$$

where  $W$  is the thickness of the tank.

The total fluid momentum is obtained by integrating equation (18). The first term in the integrand is integrated:

$$\rho \int_{-W/2}^{W/2} \int_{-R}^R \int_h^L \vec{r} \times \Delta \vec{\omega} \times \vec{r} dz dx dy = \frac{m_f}{3} (R^2 + L^2 + hL + h^2) \Delta \omega \quad (19)$$

where  $m_f$  is the total fluid mass and is equal to  $2\rho R(L-h)W$ .

The second term in the integrand of equation (18) shows the change in momentum relative to the tank created by the pressure applied at the wall  $x = -R$ :

$$-p \iiint \vec{r} \times \nabla(\phi - \phi_0) \Big|_{t=0} dz dx dy = -\frac{8\rho R^2 w}{\pi^2} \left[ 2 \cosh \frac{\pi(L-h)}{2R} + \frac{\pi h}{2R} \sinh \frac{\pi(L-h)}{2R} - 2 \right] (d_1 - E_1') \quad (20)$$

Substituting for  $d_1$  from equation (15)

$$\Delta H_f = (I_f + fG)\Delta\omega$$

where  $I_f$  is the moment of inertia of fluid about the y-axis with the fluid considered as a rigid body, that is,  $I_f = \frac{m_f}{3}(R^2 + L^2 + hL + h^2)$ , and

$$f = \frac{8\rho R^2 w}{\pi^2} \left[ 2 \cosh \frac{\pi(L-h)}{2R} + \frac{\pi h}{2R} \sinh \frac{\pi(L-h)}{2R} - 2 \right] \quad (21)$$

Substituting these relationships and equation (13) into equation (12) yields

$$\Delta\omega = \frac{\gamma}{I_t Q} \quad (22)$$

where

$$Q = 1 + \beta + \mu \left[ \frac{5 \cosh \frac{\pi(f_r - \underline{h})}{2} + \frac{3\pi \underline{h}}{2} \sinh \frac{\pi(f_r - \underline{h})}{2} - 5}{\cosh \frac{\pi(f_r - \underline{h})}{2} + \frac{\pi \underline{h}}{2} \sinh \frac{\pi(f_r - \underline{h})}{2} - 1} \right]$$

$$\beta = \frac{I_f}{I_t} = \frac{\alpha}{3} (1 + f_r^2 + f_r \underline{h} + \underline{h}^2)$$

$$\alpha = \frac{m_f}{m_t}$$

$$m_t = \frac{I_t}{R^2}$$

$$\mu = \frac{\alpha}{3} (f_r^2 + f_r \underline{h} + \underline{h}^2)$$

By substituting equation (22) into equation (17), the velocity potential function is found as a function of the impulse:

$$\phi = \left[ \left( -\frac{G\gamma}{I_t Q} + E_1' \right) \cos \sigma_1 t - \frac{\sigma_1}{\sigma_1'} (d_1' \sin \sigma_1' t_0 - e_1' \cos \sigma_1' t_0) \sin \sigma_1 t \right] \sin \frac{\pi x}{2R} \cosh \frac{\pi(L-z)}{2R} \quad (23)$$

### General Equation of Motion

After removal of the impulsive torque, the tank fluid system rotates without any additional external forces so that the system angular momentum is constant. The angular momentum after the impulse is therefore

$$H = I\omega - f(d_1 \cos \sigma_1 t + e_1 \sin \sigma_1 t)$$

where  $d_1$  and  $e_1$  are obtained from equations (15) and (16) and

$$I = I_t + I_f$$

When evaluated at  $t = 0$ , the angular momentum is found in terms of the motion to be

$$H = I\omega_+ - fd_1$$

where  $\omega_+$  is the angular velocity at the end of the impulse. Substituting for  $\omega_+$  from equation (22) gives

$$H = \frac{I\gamma}{I_t Q} + I\omega_0 - fd_1$$

Equating this formula to the angular momentum at any time  $t$  after the impulse and solving for the angular motion of the system leads to

$$\omega = \frac{\gamma}{I_t Q} + \omega_0 - \frac{fd_1}{I} (1 - \cos \sigma_1 t) + \frac{fe_1}{I} \sin \sigma_1 t \quad (24)$$

The angular motion consists of a constant  $\bar{\omega}$ , which is used to compute the slosh frequency in equation (9), and a superimposed cosine variation with time that is due to sloshing:

$$\omega = \bar{\omega} - \xi \cos \sigma_1 t - \nu \sin \sigma_1 t = \bar{\omega} - \omega_s \cos(\sigma_1 t - \psi) \quad (25)$$

where

$$\bar{\omega} = \frac{\gamma}{I_t Q} + \xi + \omega_0 \quad (26)$$

and

$$\psi = \arctan \frac{\nu}{\xi}$$

Also, the slosh coefficient is

$$\omega_s = (\xi^2 + \nu^2)^{1/2} \quad (27)$$

where

$$\xi = -\frac{f d_1}{I} = \frac{f}{I} \left( \frac{G \gamma}{I_t Q} - E_1' \right) \quad (27a)$$

and

$$\nu = \frac{f}{I} \left[ \frac{\sigma_1}{\sigma_1'} (d_1' \sin \sigma_1' t_0 - e_1' \cos \sigma_1' t_0) \right] \quad (27b)$$

In a system started from rest where  $E_1' = 0$  and  $\nu = 0$ , equation (26) can be written as

$$\bar{\omega} = \frac{\gamma}{I_t Q} + \xi$$

and equations (27) can be written as

$$\omega_s = \xi \cos \sigma_1 t$$

Therefore, equation (25) becomes

$$\omega = \frac{\gamma}{I_t Q} + \xi (1 - \cos \sigma_1 t) \quad (28)$$

## DISCUSSION AND RESULTS

From the expression for the slosh frequency given by equation (9a), a stability margin for the free surface is revealed. For example, if the thrust of the vehicle is reversed in direction and is of sufficient magnitude so that

$$-N_{Bo} \geq N_{ce} + \frac{\pi^2}{4}$$

then the motion of the free surface is unstable. Therefore, for a large tank a small thrust reversal can create instability. When the Bond number is in this region, the net force on the liquid would force it toward the base of the tank and cause a perturbation of the vehicle motion. This result suggests a limit on the magnitude of retrograde thrust on the vehicle. The limit may be exceeded to some degree if the force of impact and resulting perturbation are within a tolerance for acceptable control.

The slosh frequency during a cycle for a given fluid is found to be a function of the height of the flat equilibrium free surface of the liquid, centrifuge number, Bond number, and tank capacity. In figure 2, the product of slosh frequency  $\sigma_1$  and tank radius  $R^{3/2}$  for liquid oxygen with  $\frac{T}{p} = 4.1 \times 10^{-4} \text{ ft}^3/\text{sec}^2$  is plotted against the product of fineness ratio  $f_r$  and the ratio of residual fuel mass ratio  $\alpha$  to initial full-tank fuel mass ratio  $\alpha_{ft}$  for various centrifuge numbers and Bond numbers. As the value of the product  $f_r \frac{\alpha}{\alpha_{ft}}$  decreases below approximately 1.5 (corresponding to a fluid depth of 1.5 tank half-widths), the value of the slosh-frequency parameter  $\sigma_1 R^{3/2}$  decreases rapidly for a given tank. The rate of decrease is greater for a high Bond number at the same centrifuge number. The decrease is attributable to the increased effect of the tank top on the wave motion as  $h$  increases as the fuel is consumed. The values of the frequency parameter show that for a tank with diameter of 2 to 10 feet, a full period can take up to 20 minutes.

For maneuvers requiring short time intervals between impulse applications, the vehicle motion would be similar to a rigid body response with a slight variation due to surface displacement:

$$\omega = \frac{\gamma}{I_t Q} + \omega_0 - \nu \sigma_1 t$$

However, after repeated impulses, the coefficient  $\nu$  can increase to the point where it may become uncontrollable.

In order to minimize the sloshing, each of the components in equation (27) must be minimized separately. Examination of equations (27a) and (27b) shows that exact knowledge is required of surface velocities and displacements in order to adjust correctly the last impulse for zero sloshing.

The requirements can be simplified by applying the impulses at a time corresponding to zero surface displacement. As an illustration, consider a system

started in motion from rest by an impulse with another impulse to be applied at some later time in order to achieve some unperturbed angular rate. The components in equation (27) after the second impulse are therefore

$$\xi = \frac{f}{I} \frac{G}{I_t Q} (\gamma + \gamma_0 \cos \sigma_1' t_0)$$

$$\nu = - \frac{f}{I} \frac{\sigma_1}{\sigma_1'} \frac{G}{I_t Q} \gamma_0 \sin \sigma_1' t_0$$

One way to minimize the coefficients  $\xi$  and  $\nu$  is to divide the required impulses into two equal parts with the same sign and applied at a time corresponding to

$$t = \frac{\pi}{\sigma_1}$$

Another procedure is to apply two impulses, equal and opposite in sign, at a time corresponding to

$$t = \frac{2\pi}{\sigma_1}$$

However, such impulses call for great periods of time. During this time it could be expected that the viscosity, which has been neglected in this analysis, would damp out the sloshing effect.

An alternative to these aforementioned procedures for minimizing the sloshing is to use a two-impulse system with the initial impulse greater than required for a desired angular rate. At some later time a second impulse is applied in retrograde fashion relative to the initial impulse and with a lesser magnitude. The second impulse must be adjusted in magnitude and applied at a time

$$t = \frac{\arccos \frac{\gamma}{\gamma_0}}{\sigma_1'}$$

so that the final average angular rate is the desired rate and the coefficient  $\xi$  is zero. However, a sloshing coefficient remains due to the displacement of the free surface at the time of the last impulse. Therefore, limitations are imposed on this system to achieve a reduction in the sloshing coefficient by using an initial overshoot maneuver. The first restriction is the following relationship between the desired angular rate and the initial angular rate:

$$\bar{\omega}_0 (1 - \cos \sigma_1' t_0) = \bar{\omega}_d \quad (29)$$

where  $\bar{\omega}_d$  is the desired average angular rate of the tank.

In order to have a reduced sloshing through this initial overshoot maneuver, the following restriction on the initial angular rate must hold

$$\bar{\omega}_0^2 > \left( \frac{N_{Bo} + \frac{\pi^2}{4}}{\frac{\rho h}{\tau} R^3} \right) \frac{2}{\cos \sigma_1 t_0 (1 - \cos \sigma_1 t_0)} \quad (30)$$

Thus, for times close after the initial impulse or for times close to a quarter cycle, the qualification on  $\bar{\omega}_0$  becomes impractical due to the fact that  $\bar{\omega}_0$  would have to approach infinity. In figure 3 an example of this restriction is plotted for a tank with a fineness ratio of 2 and a radius of 4 feet and being one-half full of liquid oxygen. If  $\bar{\omega}_0$  falls above the hatched area for a given angle of  $\sigma_1 t_0$  and satisfies the condition given by equation (29), a reduction in sloshing can be obtained.

The response to an impulse is now investigated for a system initially at rest. A specific example of the angular response of a fluid tank system started from rest by an impulsive torque is presented in figure 4. The tank is considered to be one-half full of liquid oxygen and to have a pitching moment of inertia of 720 slug-feet<sup>2</sup>, a half-width of 4 feet, a fineness ratio of 2, and an imposed acceleration field "a" due to thrust of  $5.15 \times 10^{-5}$  ft/sec<sup>2</sup>. The impulse is of sufficient magnitude to produce an average angular velocity  $\bar{\omega}$  of 0.0025 radian/sec, a slosh coefficient  $\omega_s$  of 0.0014 radian/sec, and a  $\Delta\omega$  of 0.0011 radian/sec which is the initial angular velocity of the system since the tank is started from rest. These responses are normalized with respect to the impulse and are plotted against the ratio  $\alpha$  (fluid mass to tank mass) for different tank fineness ratios  $f_r$  and different values of the ratio  $\alpha_{ft}$  (full-tank fluid mass to tank inertial mass). The vehicle considered has a pitching moment of inertia of 720 slug-feet<sup>2</sup>. The curves in all the responses are not carried out to the full-tank values because the theory calls for a free surface at all times, which cannot be achieved with a full tank.

In figure 5 the change in angular velocity per unit impulse as calculated from equation (22) is shown. For fineness ratios of 3 and 4 the response remains approximately constant as  $\alpha$  decreases until values of about 0.2 are reached, at which time there is a rapid increase converging to the rigid-body response of the vehicle. Smaller values of fineness ratio produce a larger response for all tanks and a more gradual increase with decreasing amounts of available fuel. For tanks with lesser capacities, the responses are greater at the same value of  $\alpha$ . Thus, for a tank with a fineness ratio of 1, the increase in response with decreasing  $\alpha$  resembles an exponential curve; whereas, for a fineness ratio of 4, the response remains approximately constant until the previously mentioned value of  $\alpha$  of about 0.2 is reached.

In figure 6 the average angular velocity is shown as computed from equation (26). The response is similar to the change in angular velocity shown in figure 5, except for the addition of the slosh coefficient. The convergence to a rigid-body response as the tank empties is more gradual for tanks with small

capacities. For any tank capacity, the smaller fineness ratios have a more gradual convergence to a rigid-body response.

The slosh coefficient per unit impulse is shown in figure 7 as computed from equation (27). For values of  $\alpha_{ft}$  of 1.25 and 1.00, the maximum values of  $\omega_s/\gamma$  remain approximately the same, with the peak values occurring at smaller values of  $\alpha$  for larger fineness ratios. In general, the values for a tank capacity of 1.25 and 1.00 increase for decreasing fineness ratios. Also, as the tank capacities decrease, the general range of values becomes greater. However, as the tank empties there is a crossover point where the values for the larger fineness ratios become larger than the values for the smaller fineness ratios. As the tank capacities decrease, the crossover point occurs at increasingly larger values of  $\alpha$  until, as shown in figure 7(d), with  $\alpha_{ft} = 0.50$  the values of  $\omega_s/\gamma$  for a fineness ratio of 2 are greater over the entire range than for a fineness ratio of 1. It should be noted that the results are not applicable when the fluid level nears the top of the tank because the pressure over the top of the tank has been assumed to be negligible in the discussion of initial conditions.

In figure 8, the slosh coefficient per average angular velocity  $\omega_s/\bar{\omega}$  is plotted against  $\alpha$ . This figure shows that the larger the fineness ratio and the larger the tank capacity, the greater is the ratio of slosh coefficient to angular velocity. The ratio decreases slowly as the fluid level in the tank decreases until values of  $\alpha$  of about  $0.25\alpha_{ft}$  are reached. At this time, the decrease becomes faster and converges to the condition of an empty tank and rigid-body response.

## CONCLUSIONS

The equation of motion has been derived for the two-dimensional angular rotation of a vehicle containing a large amount of an irrotational, incompressible, and inviscid liquid fuel with a flat equilibrium free surface contained in a rectangular tank. The derivation includes a determination of a velocity potential function and frequency of free-surface oscillation generated by an impulsive torque. Analysis of these equations reveals the following conclusions:

1. The free-surface motion of a given liquid is a stable oscillation until a negative inertial loading factor due to a thrust reversal exceeds the sum of a centrifugal-force loading factor and  $\frac{\pi^2}{4}$ . Under these conditions this free-surface motion diverges. For large tanks, only small thrust reversals are required to create this divergence.

2. The frequency of the free-surface motion for a given liquid in a given-size tank under constant inertial loading remains approximately constant until the depth of the fluid in the tank reaches approximately 1.5 tank half-widths. At this time there is a rapid convergence to zero.

3. The vehicle angular motion can be described as a constant average angular velocity with a superimposed cosine variation with time whose amplitude or slosh coefficient is a function of the amount of fluid in the tank, the magnitude of the applied impulse, and the time of its application in the cycle previous to the applied impulse. The change in the angular velocity during the time of the impulsive torque is a function of the last impulse, with the resulting angular velocity being the initial angular velocity of the new cycle.

4. The effect of sloshing on the angular motion of the vehicle can be minimized by utilizing two equal impulses with the second impulse applied out of phase with the sloshing so as to cancel the angular momentum of the fluid. A disadvantage to this method is that large periods of time are required between impulses. Another method is to utilize two impulses that are not equal in magnitude or sign and applied at shorter intervals of time. This method can possibly reduce the sloshing effect to within tolerable limits.

5. The change in angular velocity and average angular velocity due to an impulse are greater for small fineness ratios, with a gradual increase in the response as the tank empties until there is a small amount of fluid remaining in the tank. Below these amounts of fluid there is a rapid convergence to the rigid-body response of the vehicle.

6. In general, the values of the slosh coefficient are greater for smaller fineness ratios for given amounts of remaining fluid. These values also increase with decreasing values of the ratio of initial fuel mass to vehicle mass. As the tank empties, maximum values of slosh coefficient are reached that are about the same for all fineness ratios, with these maximum values occurring at larger values of remaining fluid for smaller values of fineness ratio. Below these amounts of remaining fluid, the slosh coefficient decreases toward zero with the rate of decrease greater for larger values of fineness ratio.

7. The ratio of slosh variation to average angular velocity is greater for large fineness ratios and for larger initial tank capacities. As the tank empties, there is a gradual decrease in this ratio; however, when a low amount of fluid remains, there is a rapid convergence to zero.

Langley Research Center,  
National Aeronautics and Space Administration,  
Langley Station, Hampton, Va., March 5, 1964.

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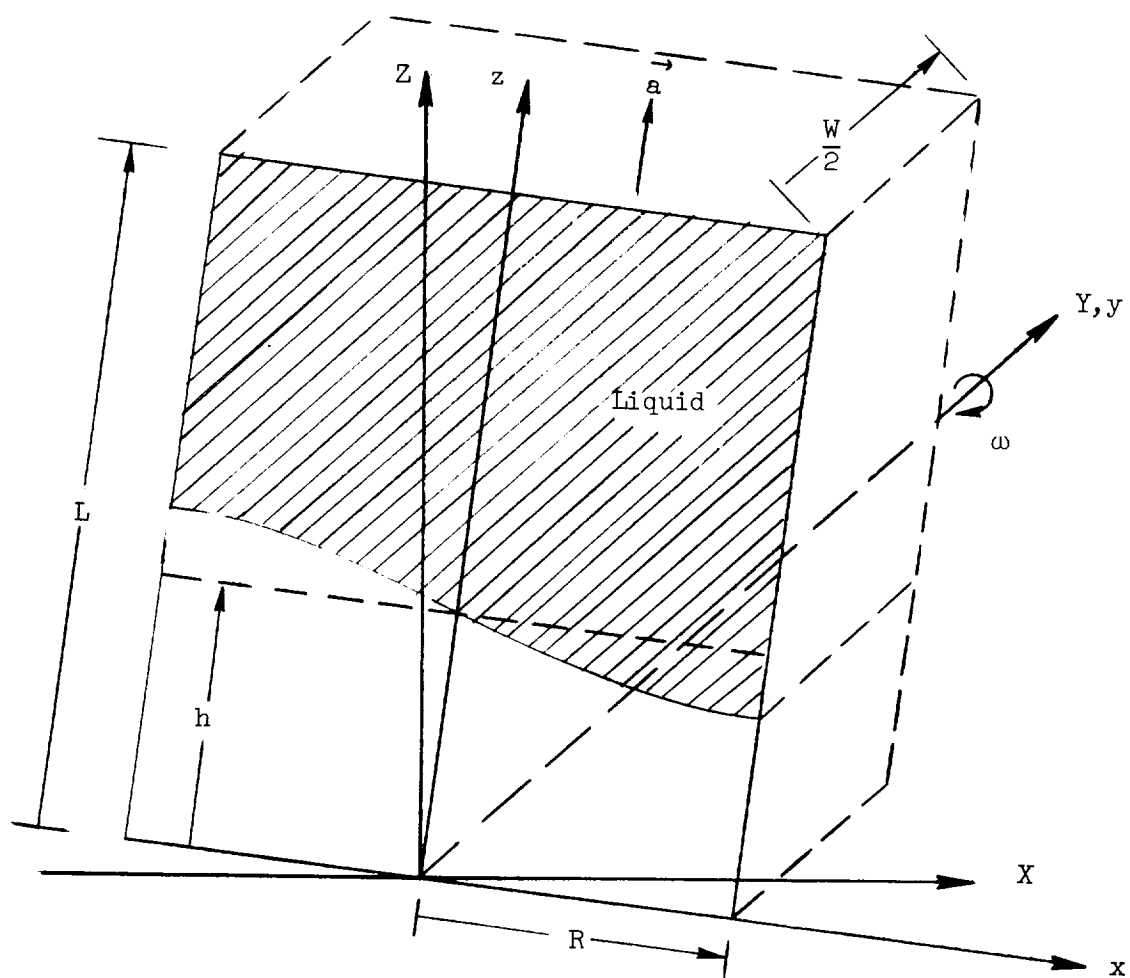
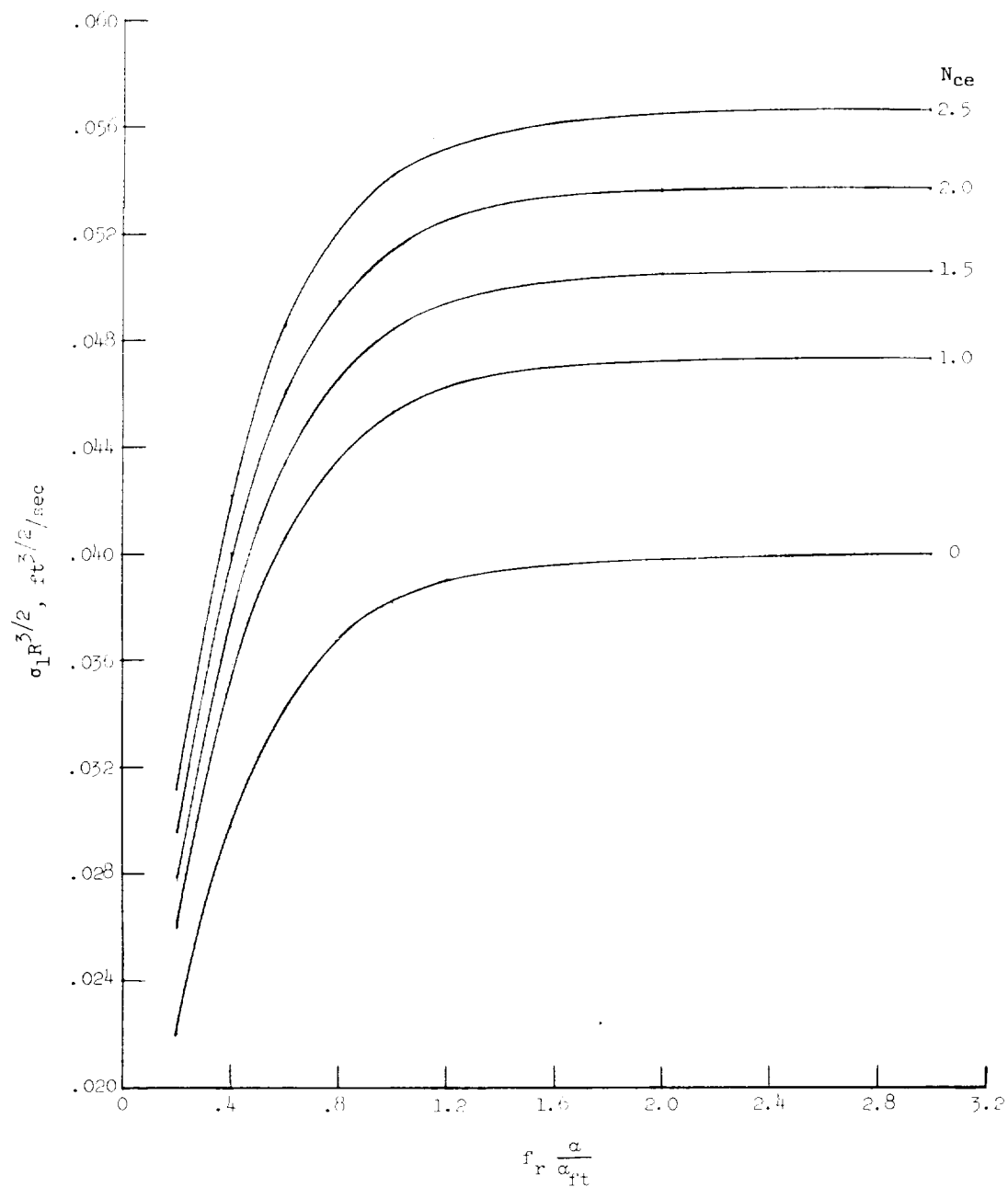
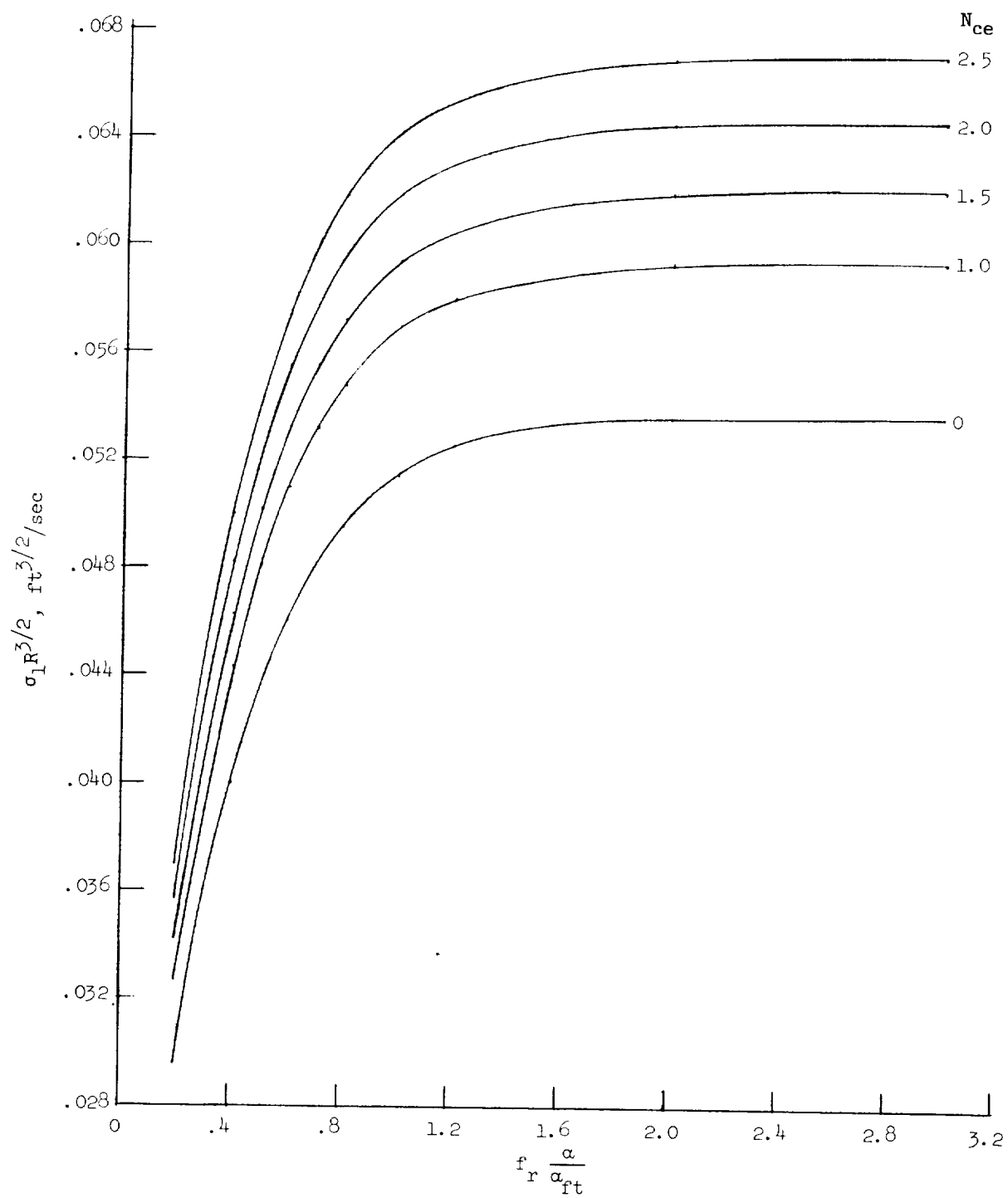


Figure 1.- Tank and fluid configuration. Arrows indicate positive directions.



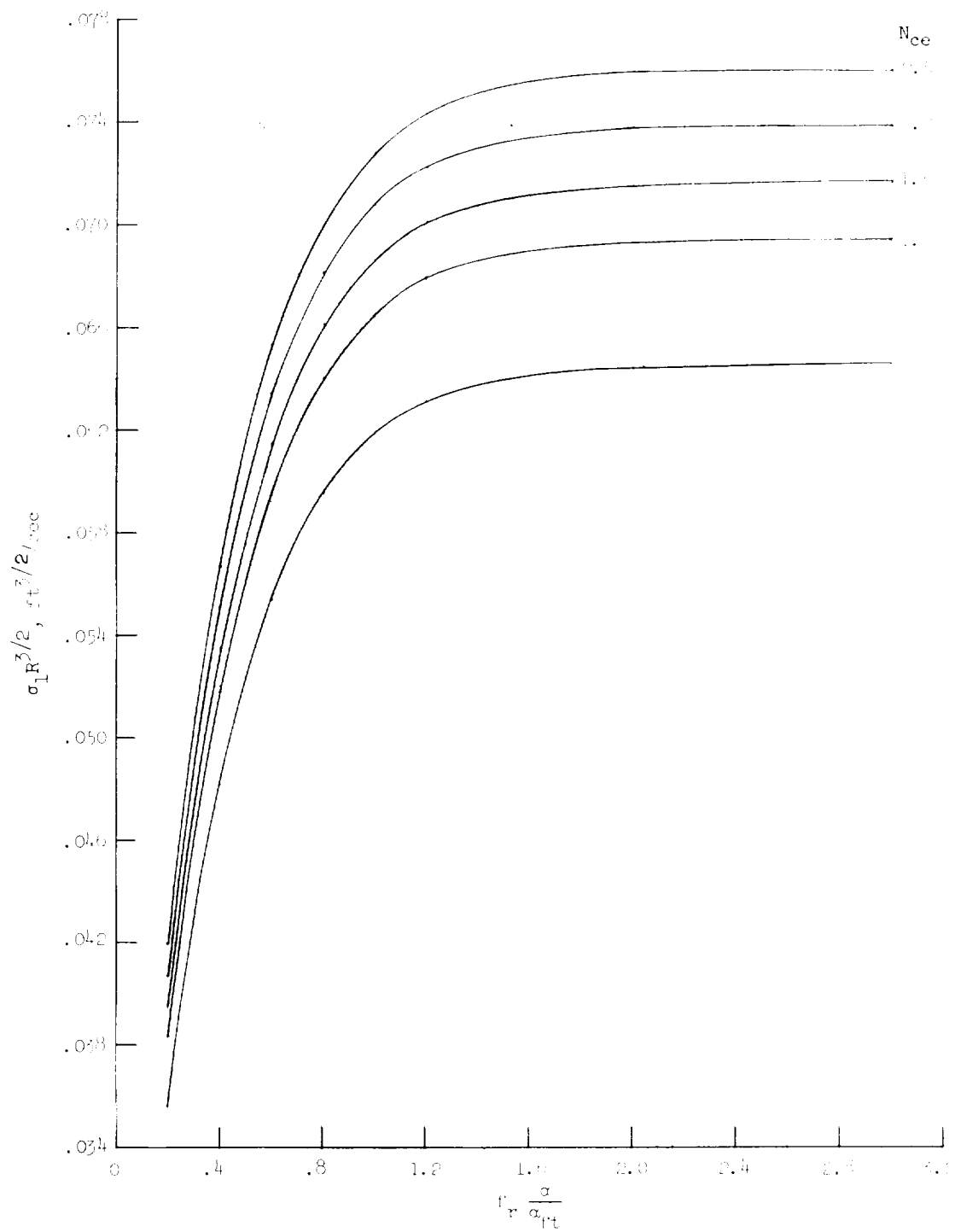
(a)  $N_{Bo} = 0$ .

Figure 2.- Product of slosh frequency and tank radius  $(\sigma_1 R^{3/2})$  plotted against product of fineness ratio and ratio of residual fuel mass ratio to initial tank fluid mass ratio  $(f_r \frac{a}{a_{ft}})$  for various values of centrifuge numbers and at different Bond numbers.  $\frac{I}{\rho} = 4.1 \times 10^{-4} \text{ ft}^3/\text{sec}^2$ .



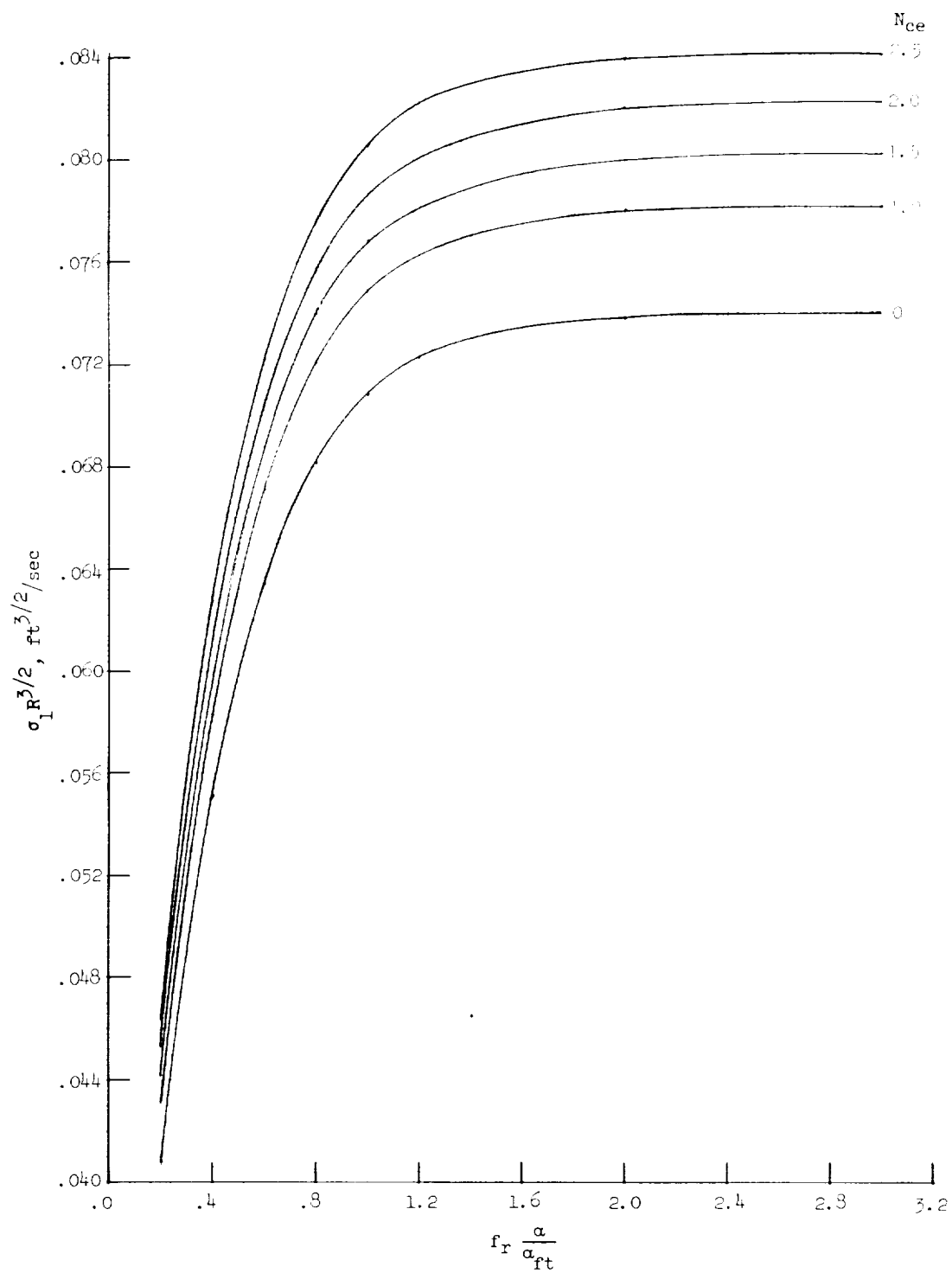
(b)  $N_{Bo} = 2.$

Figure 2.- Continued.



(c)  $N_{B0} = 4$ .

Figure 2.- Continued.



(d)  $N_{Bo} = 6.$

Figure 2.- Concluded.

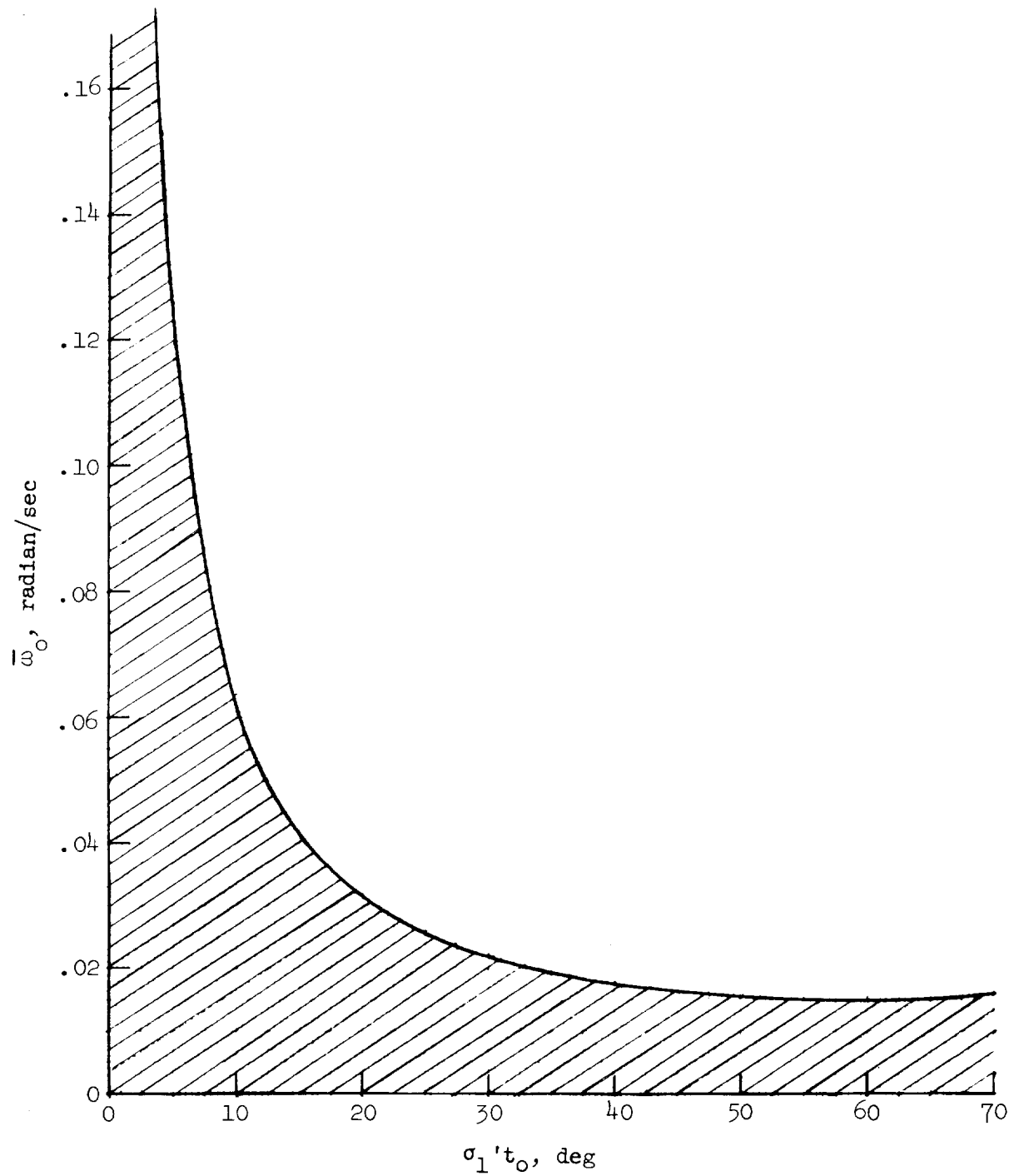


Figure 3.- Limit on initial average angular velocity to achieve a reduction in sloshing by utilizing an initial overshoot impulse maneuver.  $R = 4$  feet;  $\frac{I}{\rho} = 4.1 \times 10^{-4} \text{ ft}^3/\text{sec}^2$ ;  $f_r = 2$ ;  $\alpha = 0.5$ ;  $N_{B0} = 2$ .

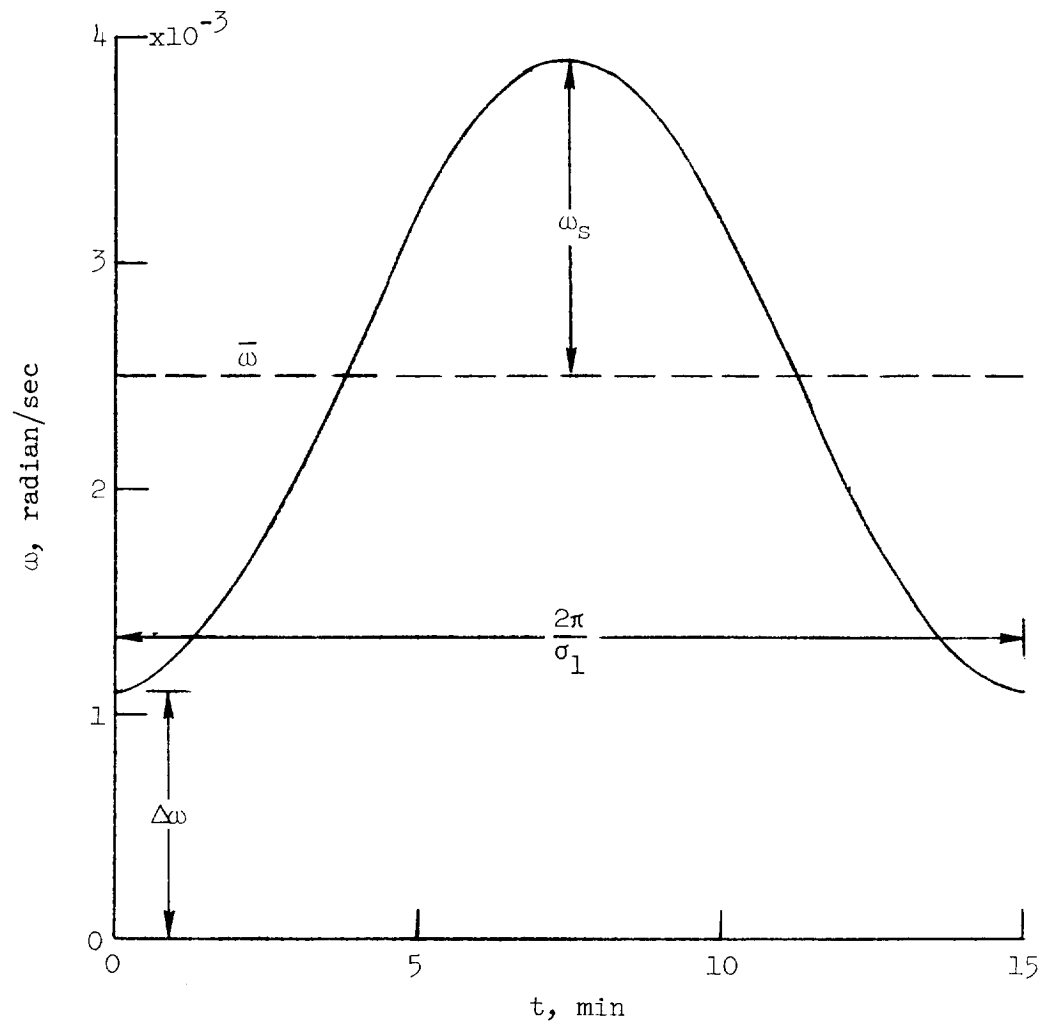
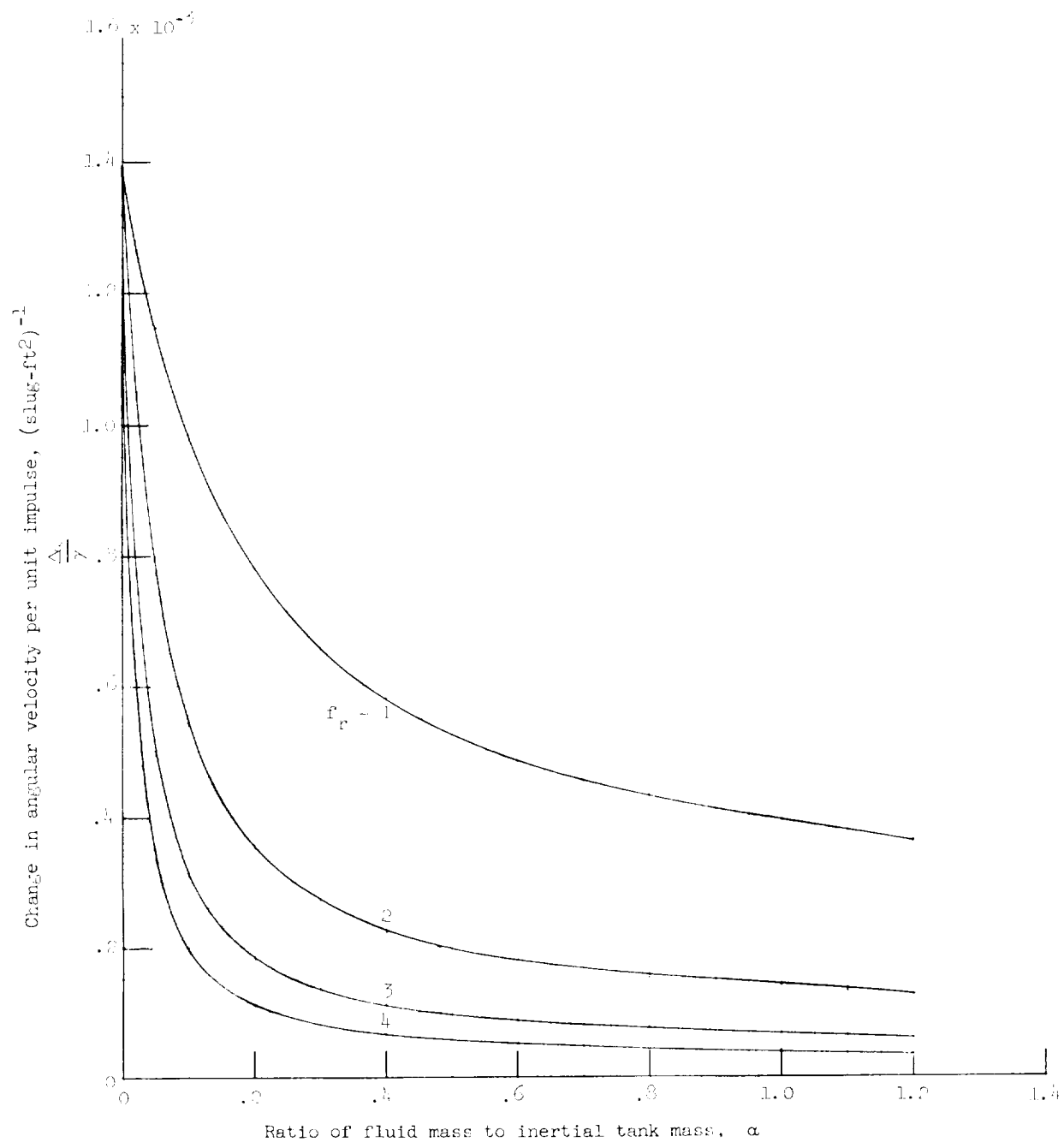


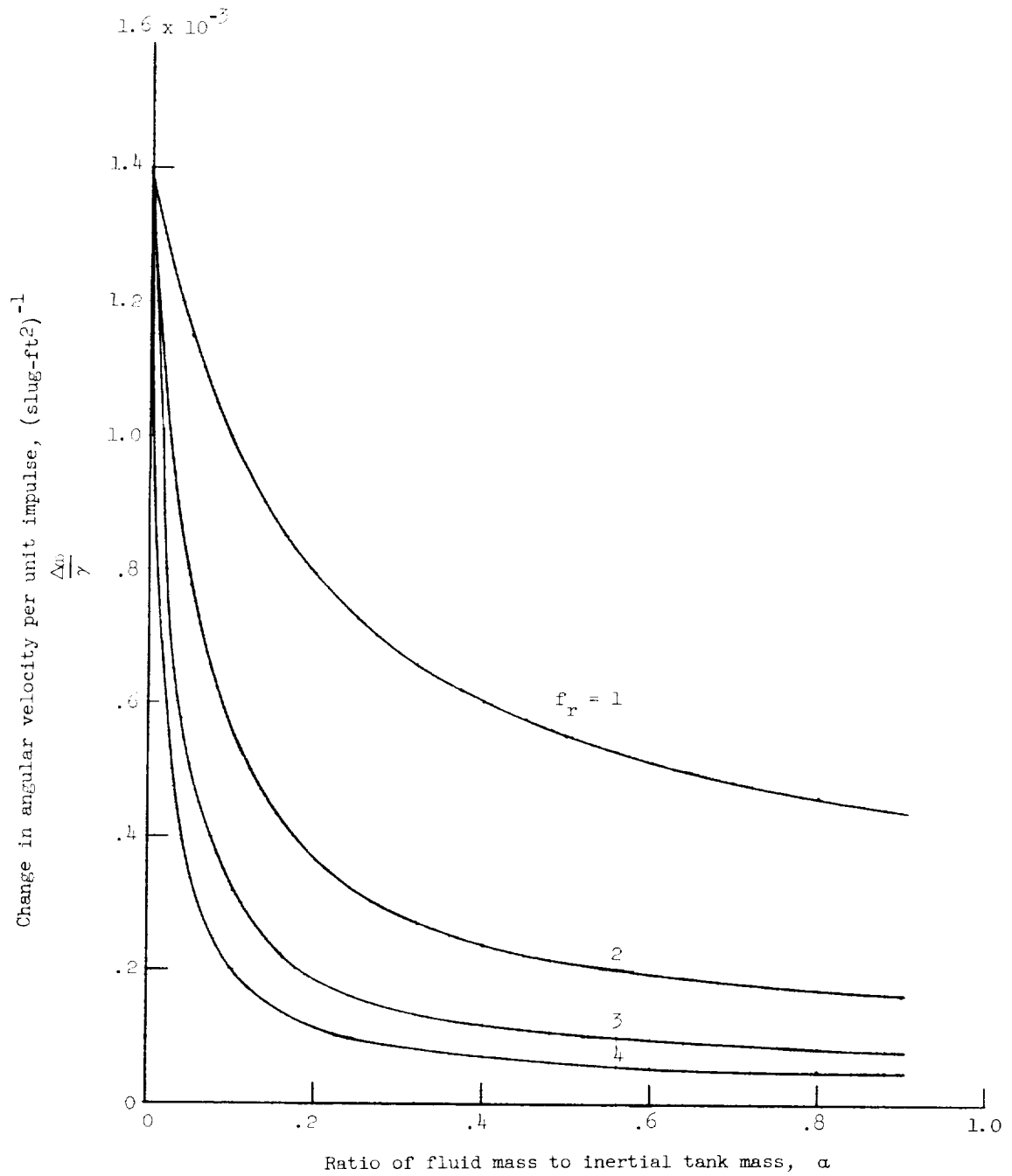
Figure 4.- Angular motion of a vehicle started from rest by an impulsive torque.

$$I_t = 720 \text{ slug-feet}^2; f_r = 2; R = 4 \text{ feet}; \alpha = 0.5.$$



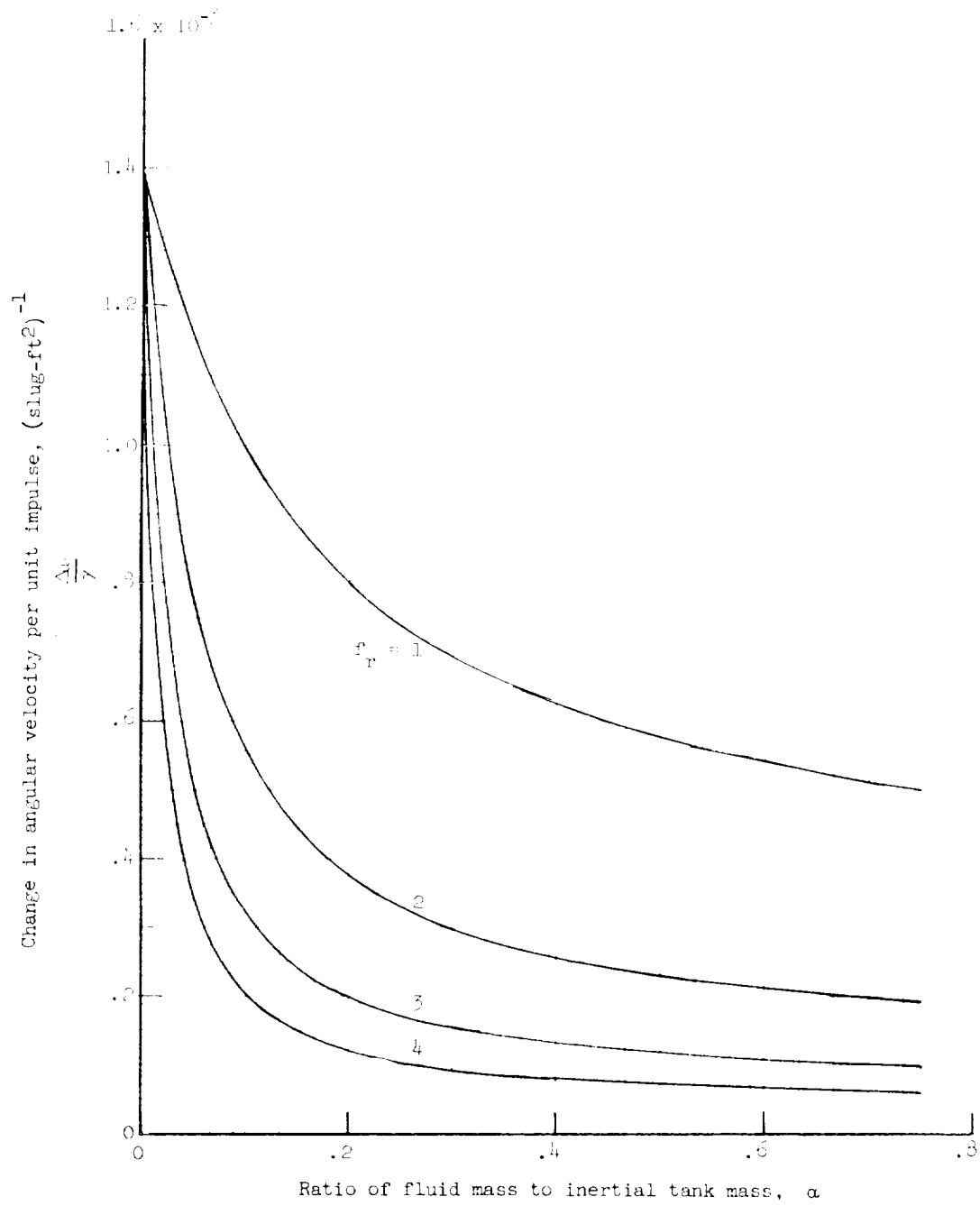
(a)  $\alpha_{ft} = 1.25$ .

Figure 5.- Change in angular velocity per unit impulse plotted against ratio of fluid mass to tank mass for various fineness ratios and at different values of  $\alpha_{ft}$  (the ratio of full-tank fluid mass to tank inertial mass).



(b)  $a_{ft} = 1.00$

Figure 5.- Continued.



(c)  $\alpha_{rt} = 0.80$ .

Figure 5.- Continued.

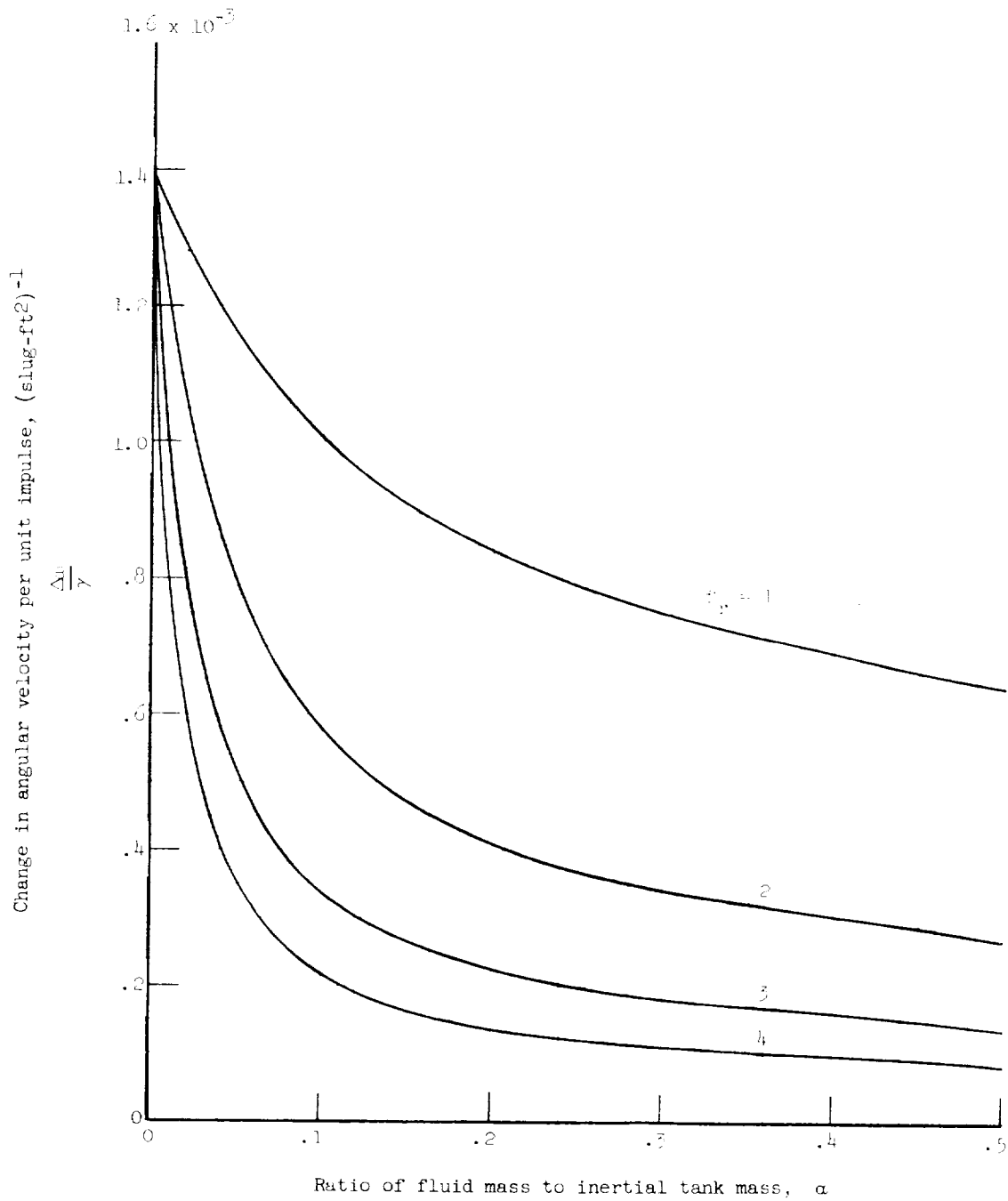
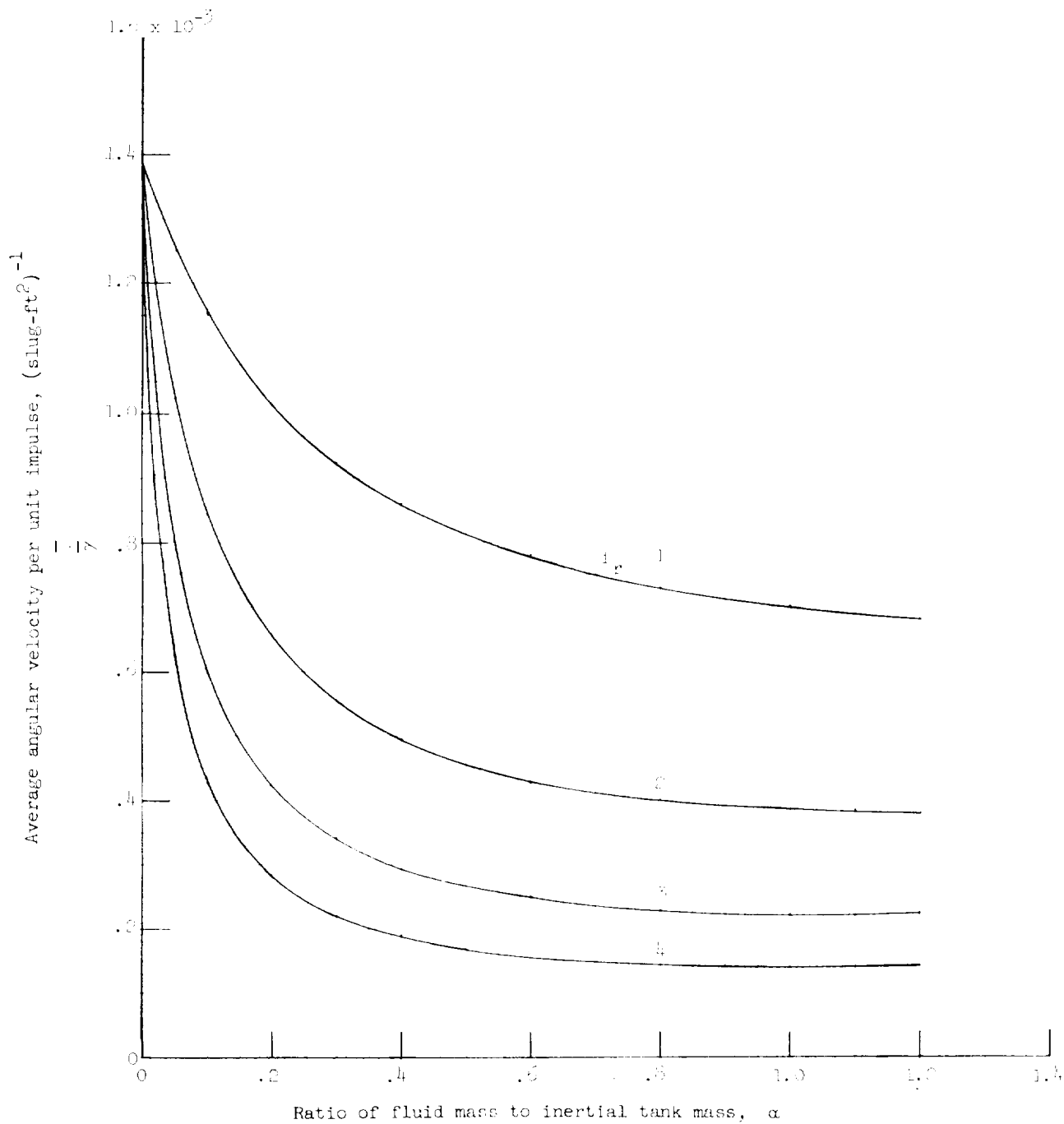
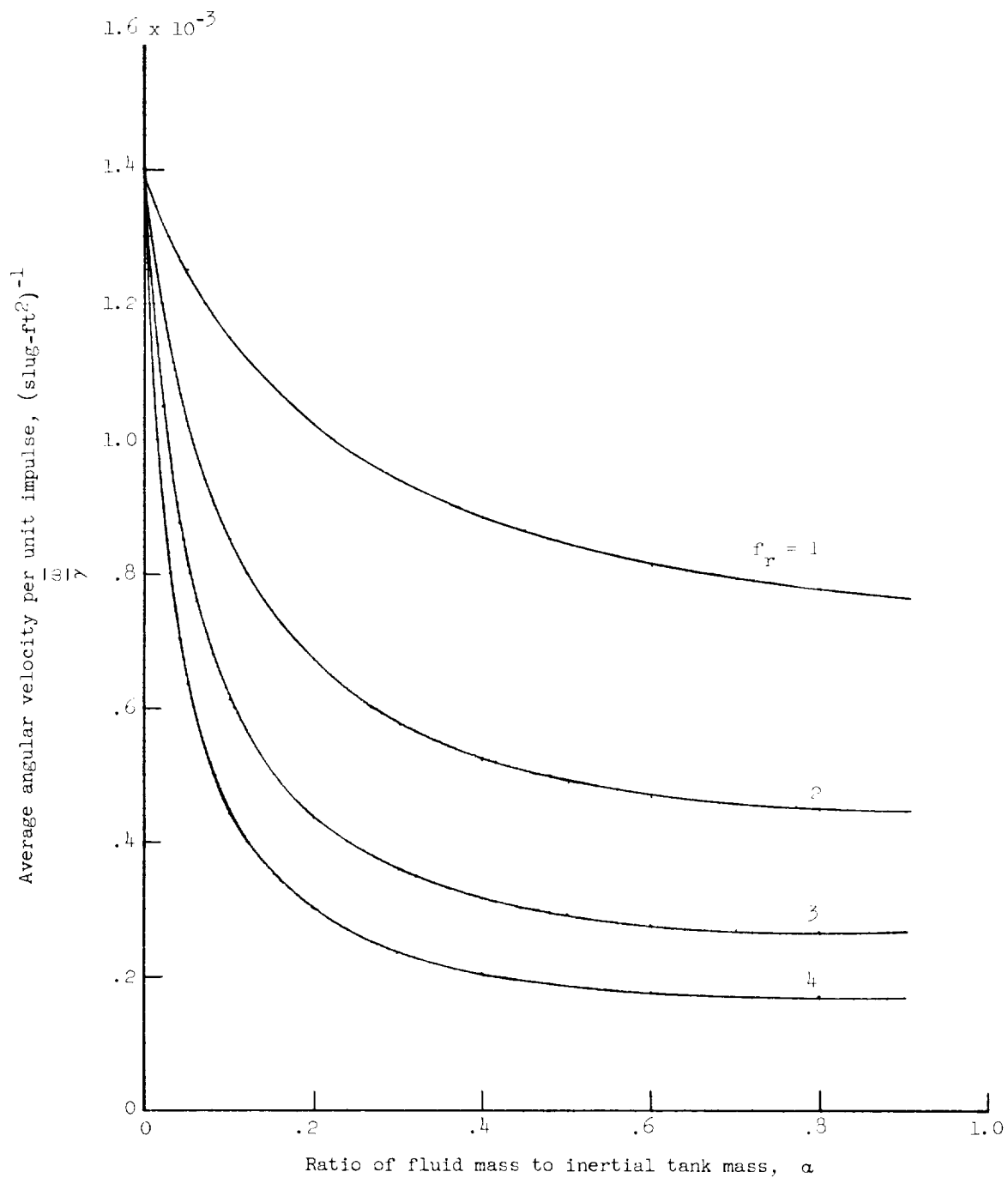


Figure 5.- Concluded.



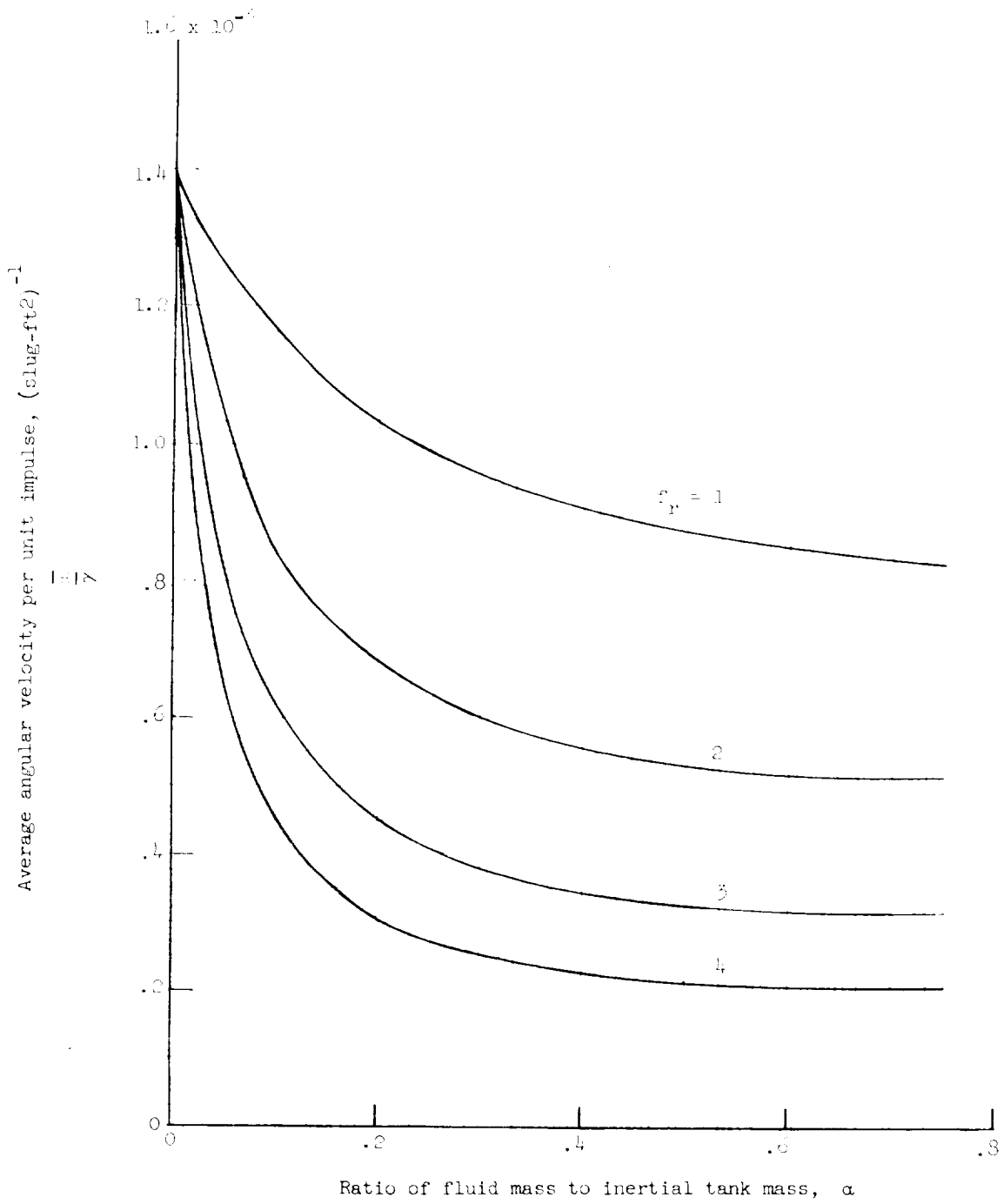
(a)  $\alpha_{ft} = 1.25$ .

Figure 6.- Average angular velocity per unit impulse plotted against ratio of fluid mass to tank mass for various fineness ratios and at different values of  $\alpha_{ft}$  (the ratio of full-tank fluid mass to tank inertial mass).



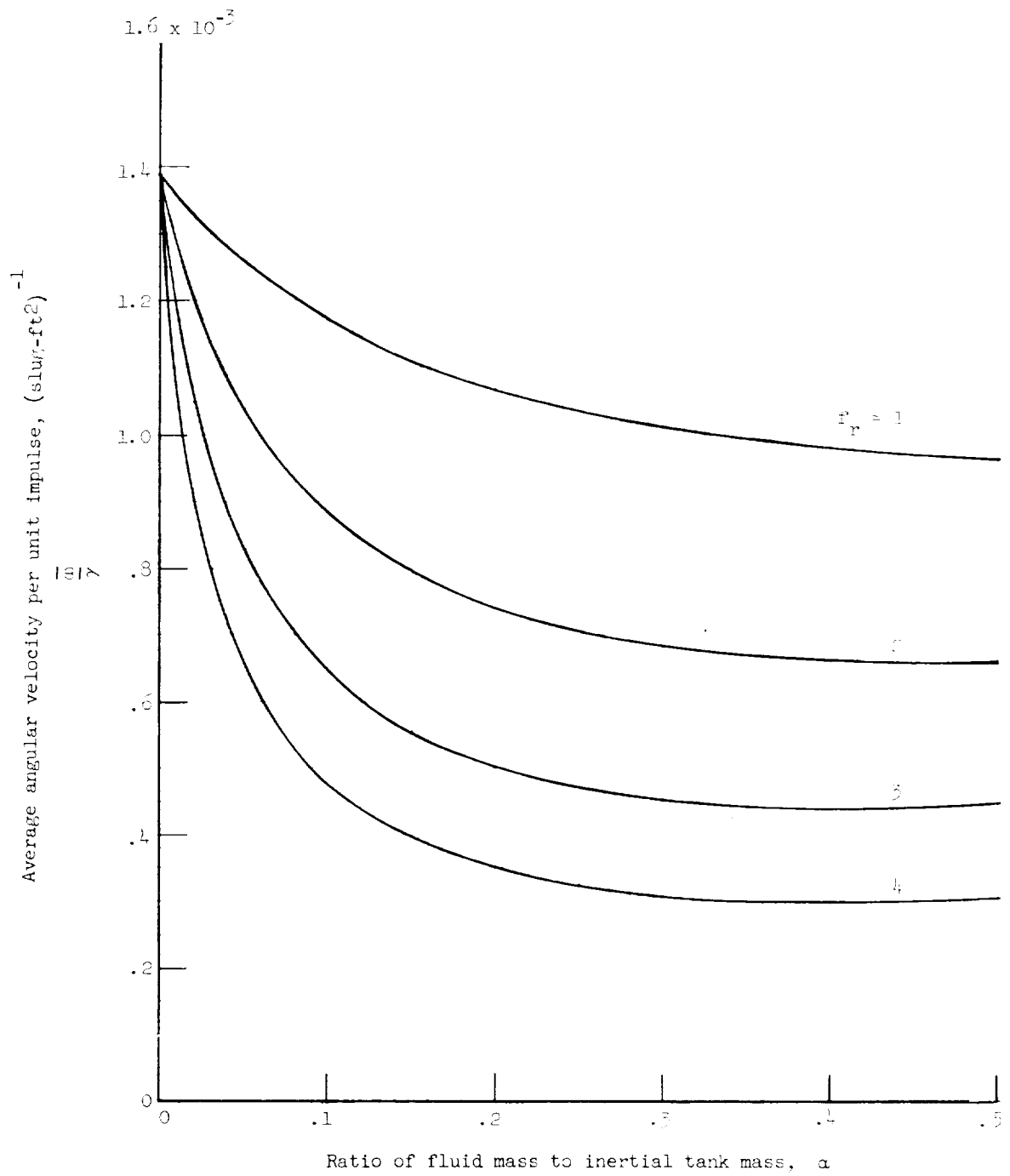
(b)  $\alpha_{ft} = 1.00$ .

Figure 6.- Continued.



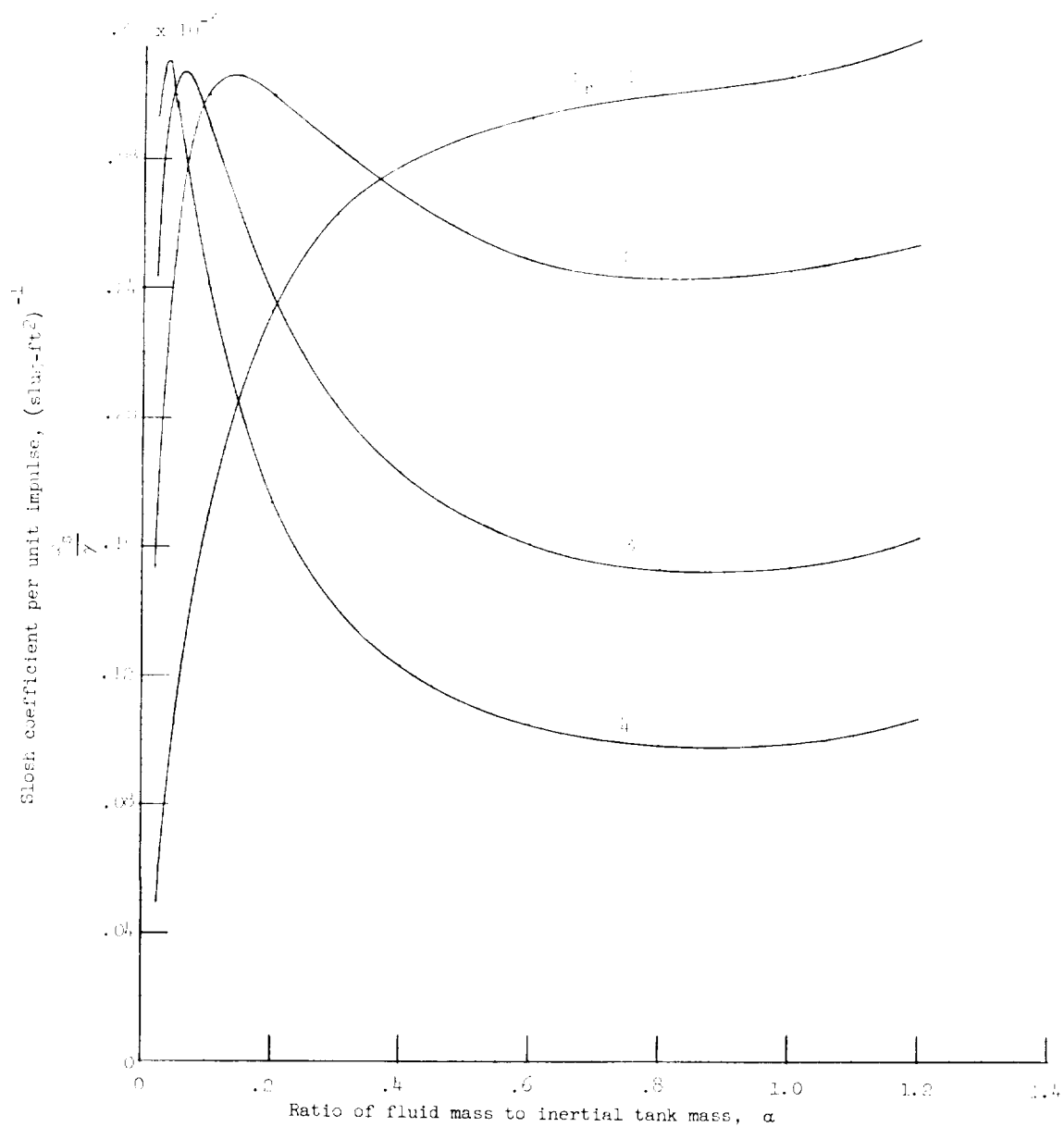
(c)  $\alpha_{rt} = 0.80$ .

Figure 6.- Continued.



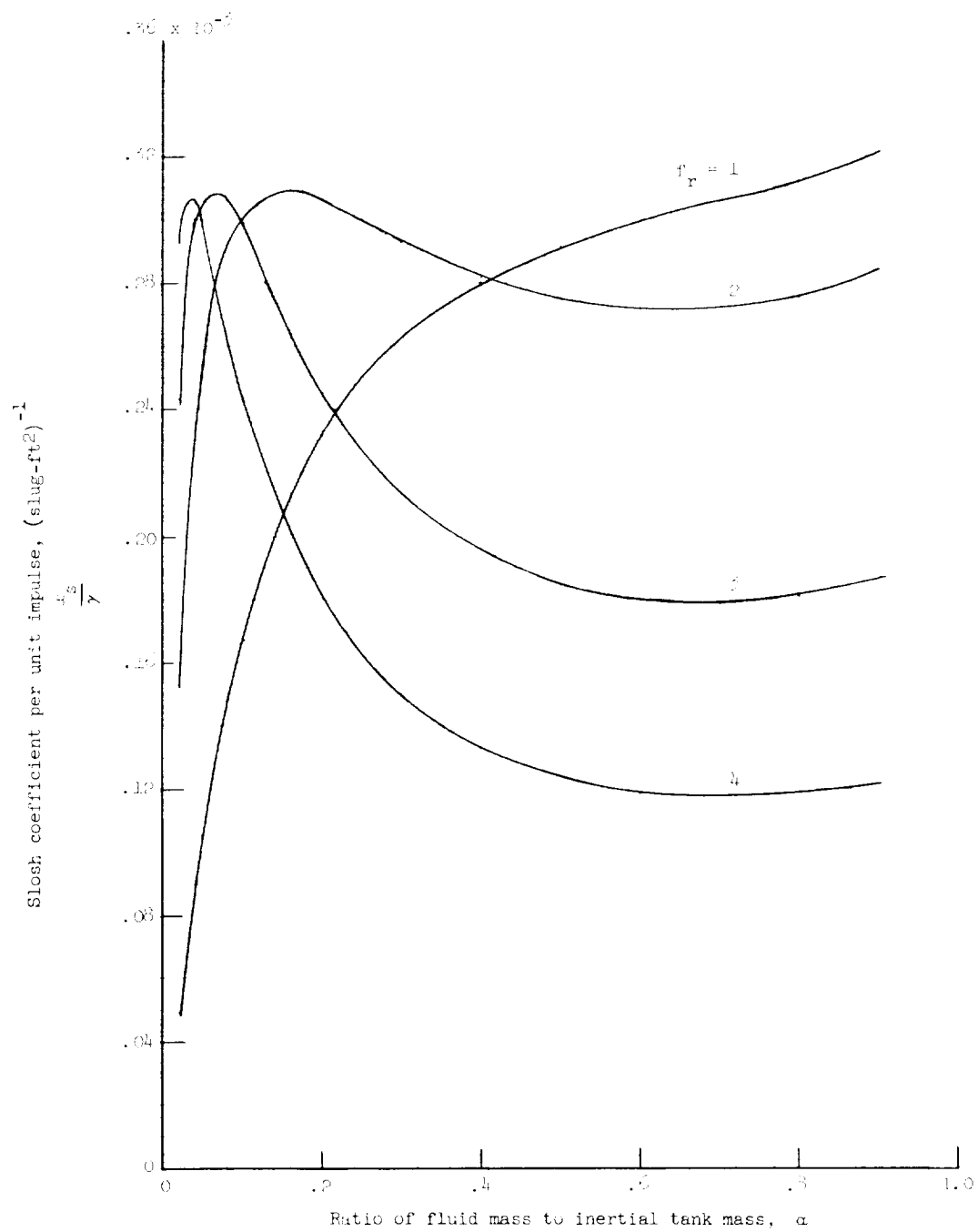
(d)  $\alpha_{ft} = 0.50$ .

Figure 6.- Concluded.



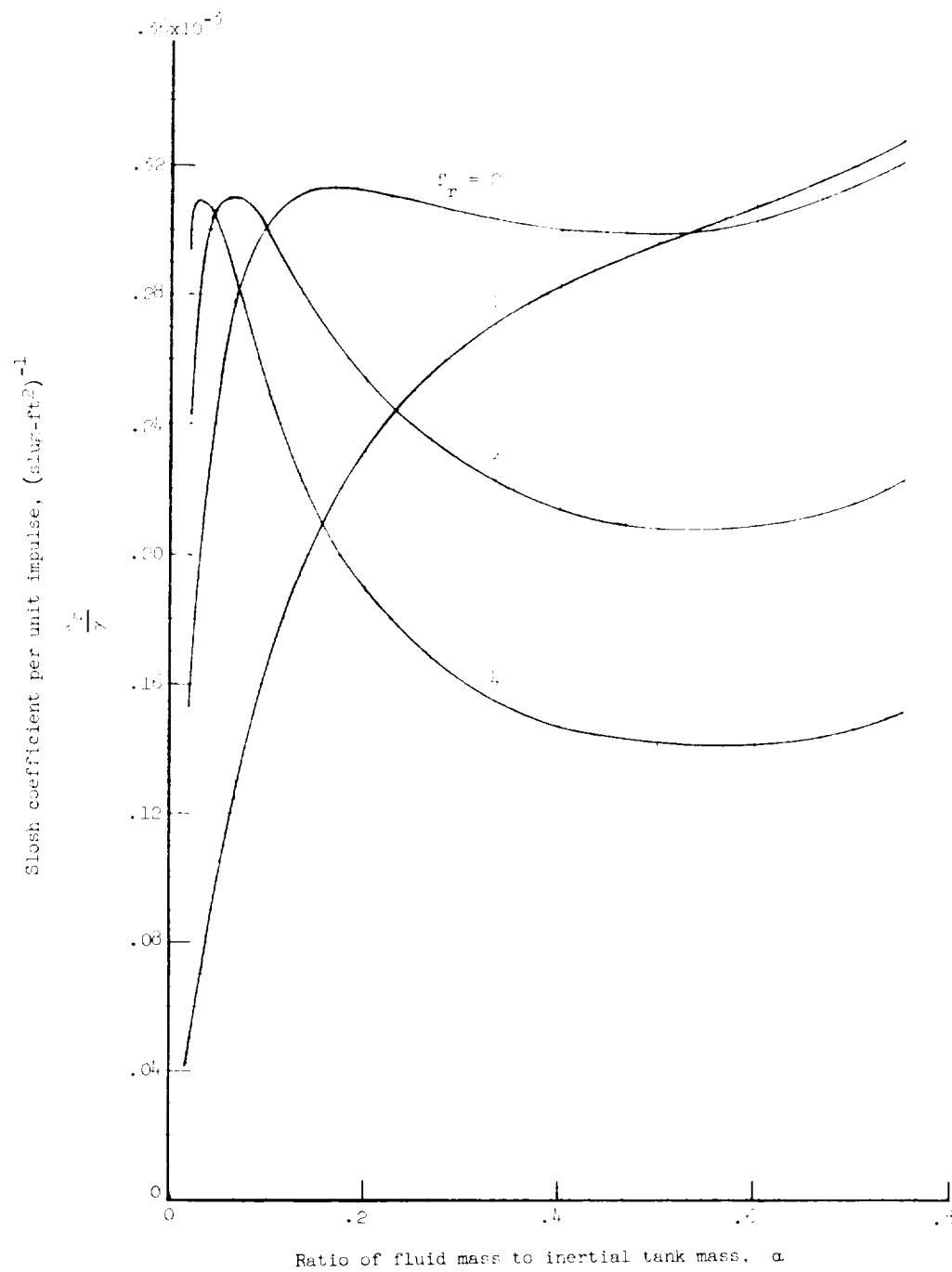
(a)  $a_{ft} = 1.25$ .

Figure 7.- Slosh coefficient per unit impulse plotted against ratio of fluid mass to tank mass for various fineness ratios and at different values of  $a_{ft}$  (the ratio of full-tank fluid mass to tank inertial mass).



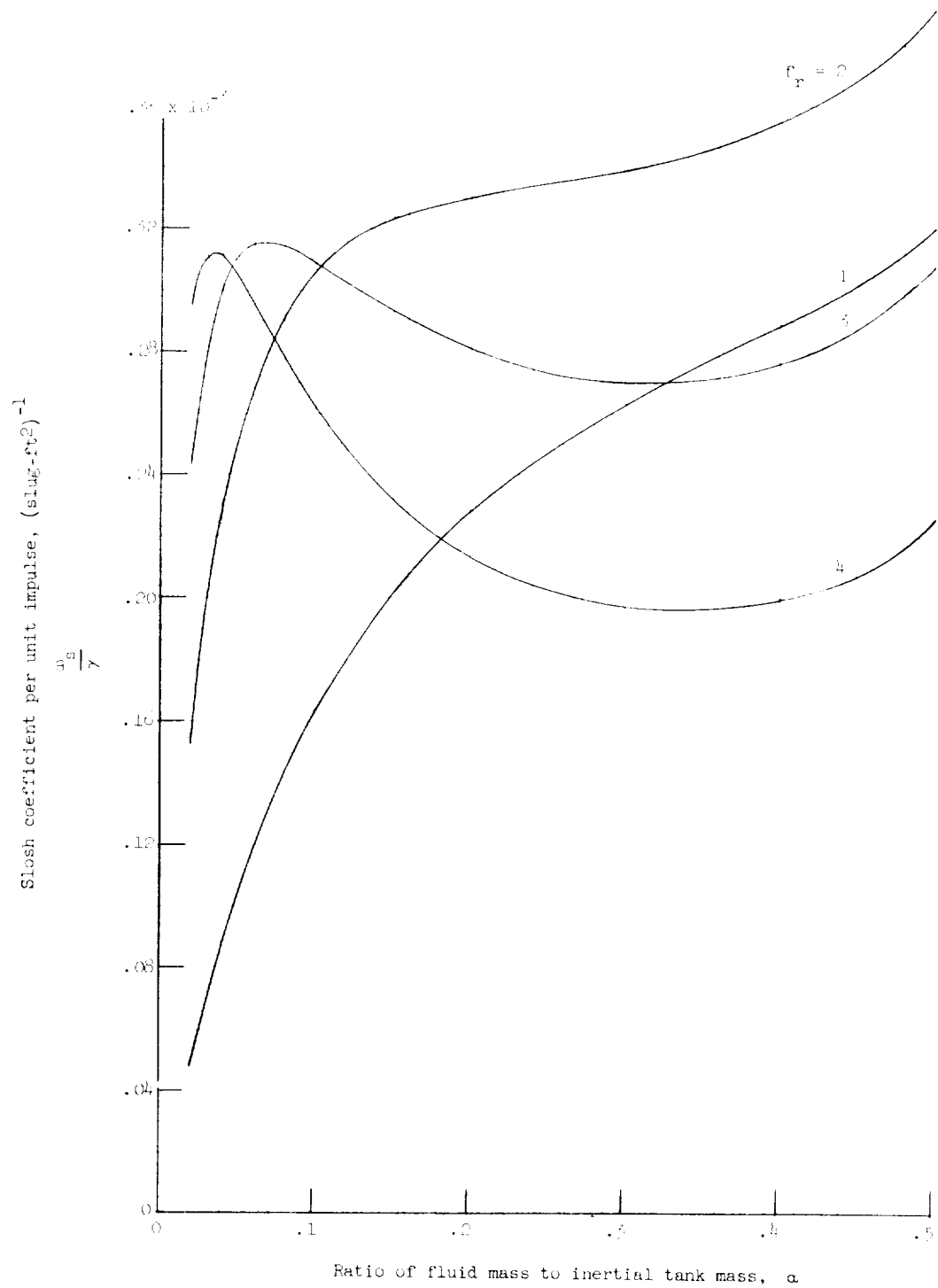
(b)  $\alpha_{pt} = 1.00$ .

Figure 7.- Continued.



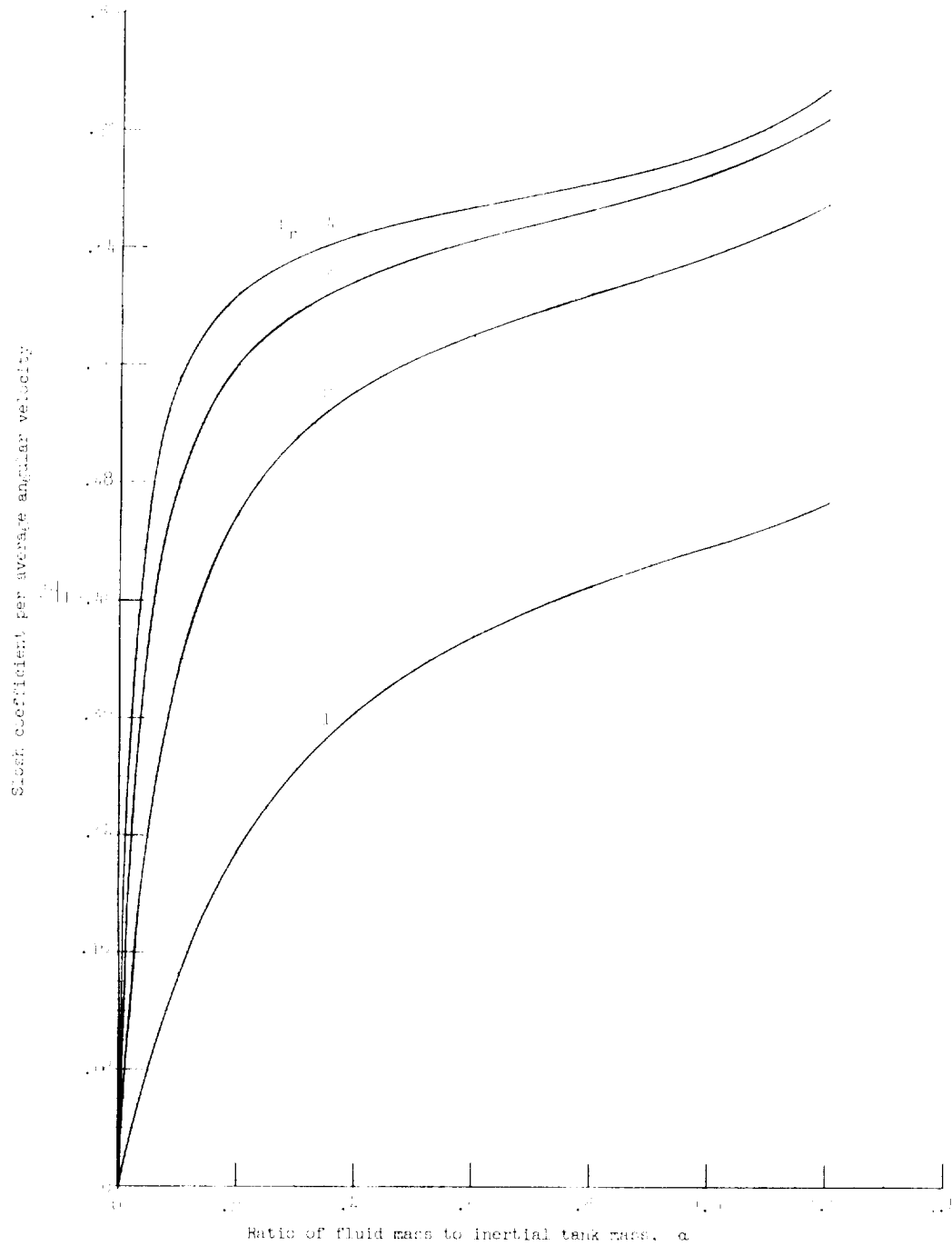
(c)  $\alpha_{ft} = 0.80$ .

Figure 7.- Continued.



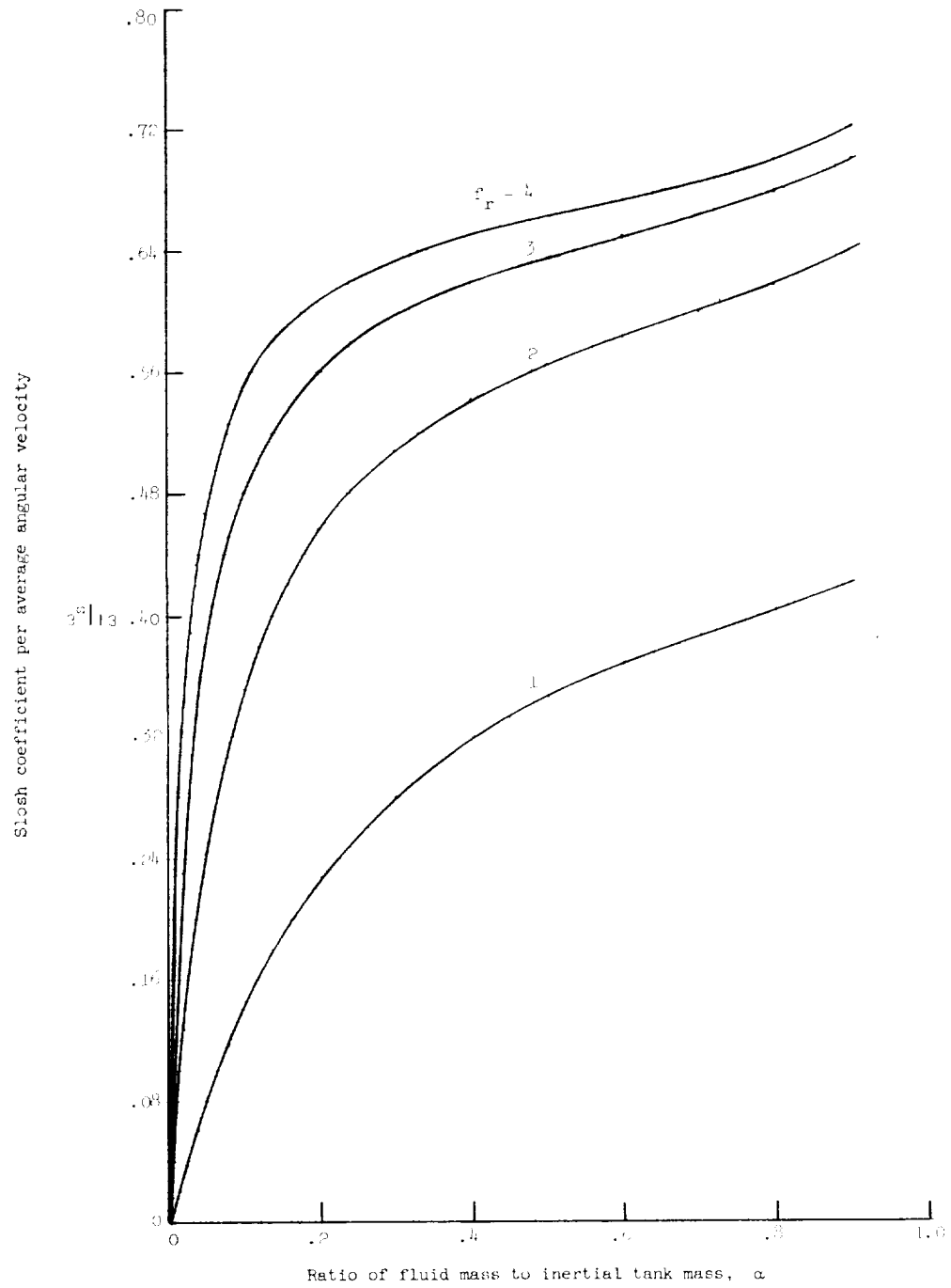
(d)  $\alpha_{rt} = 0.50$ .

Figure 7.- Concluded.



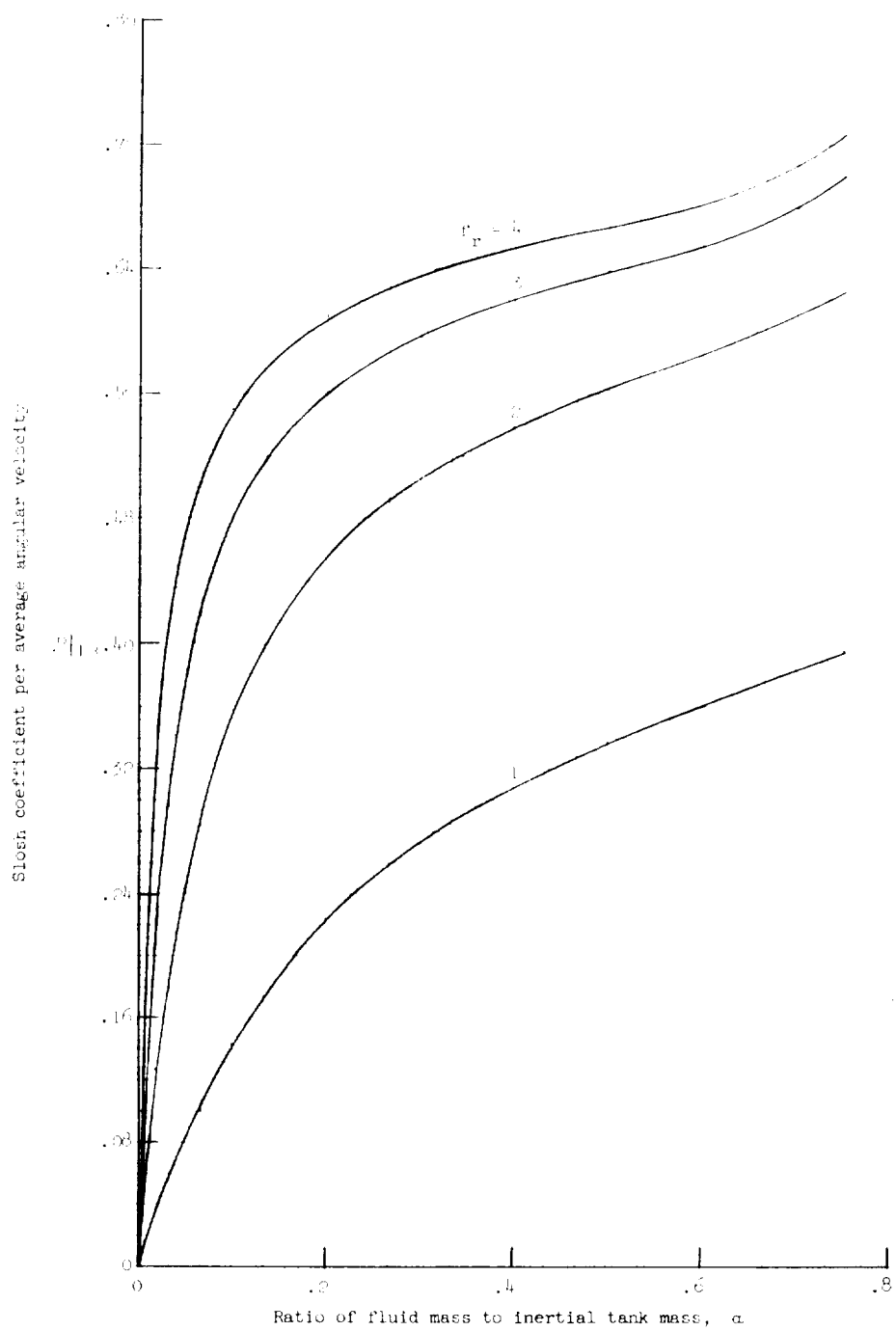
(a)  $\alpha_{pt} = 1.25$ .

Figure 8.- Slosh coefficient per average angular velocity plotted against ratio of fluid mass to tank mass for various fineness ratios and at different values of  $\alpha_{pt}$  (the ratio of full-tank fluid mass to tank inertial mass).



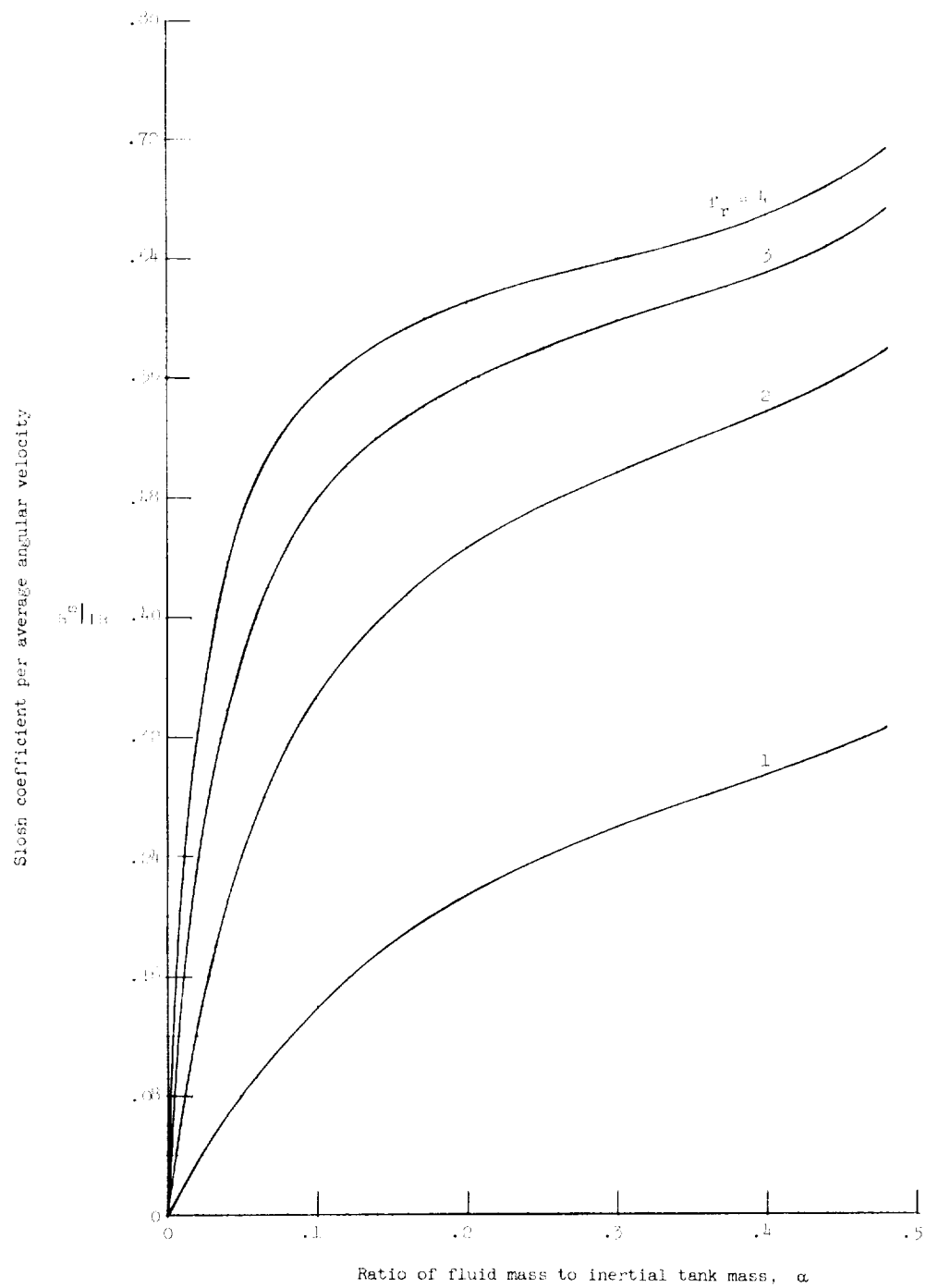
(b)  $\alpha_{rt} = 1.00$ .

Figure 8.- Continued.



(c)  $a_{ft} = 0.80$ .

Figure 8.- Continued.



(d)  $\alpha_{rt} = 0.50$ .

Figure 8.- Concluded.



<p>NASA TN D-2336 National Aeronautics and Space Administration. PRELIMINARY ANALYSIS OF VARIATION OF PITCH MOTION OF A VEHICLE IN A SPACE ENVIRON- MENT DUE TO FUEL SLOSHING IN A RECTANGU- LAR TANK. Donald G. Eide. June 1964. 48p. OTS price, \$1.25. (NASA TECHNICAL NOTE D-2336)</p> <p>The equation of motion is derived for the angular rotation of a space vehicle due to an impulsive torque about the pitch axis while thrusting and partly filled with an inviscid, incompressible, and irrotational fluid in a rectangular tank. The natural frequency of the free-surface motion is presented and the limit of thrust reversal for stability of the free surface is established. Methods of minimizing the effect of the fluid motion by proper timing and adjustment of the magnitude of the impulsive torque are discussed. The angular response of a vehicle to a unit impulsive torque is presented as a function of the inertial loading, tank size, and the amount of fluid in the tank.</p>	<p>I. Eide, Donald G. II. NASA TN D-2336</p>
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